Symmetric Powers and Eilenberg Maclane Spectra

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Abstract

We filter the equivariant Eilenberg Maclane spectrum $H\underline{\mathbb{F}}_p$ using the mod p symmetric powers of the equivariant sphere spectrum, $\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma^{\infty G}S^0)$. When G is a p-group, we show that the layers in the filtration are the Steinberg summands of the equivariant classifying spaces of $(\mathbb{Z}/p)^n$ for $n = 0, 1, 2, \ldots$. We show that the layers of the filtration split after smashing with $H\underline{\mathbb{F}}_p$. Along the way, we produced a general computation of the geometric fixed points of $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{F}}_p$ by using symmetric powers.

1 Introduction

For any abelian group A and a pointed space X let $A \otimes X$ be the space of finite A-linear combinations of points on X, with addition given by concatenation and the basepoint treated as 0. For any connected, pointed space X, the infinite symmetric power $\mathrm{Sp}^{\infty}(X)$ is weakly equivalent to $\mathbb{Z} \otimes X$, and so we write $\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(X) := \mathbb{Z}/p \otimes X$ for the mod p symmetric powers.

The Eilenberg–Maclane space $K(A, \ell)$ is given by the space $A \otimes S^{\ell}$ ([15]). Stabilizing in ℓ gives the Eilenberg-Maclane spectrum HA as the spectrum of A-linear combinations of points on the sphere spectrum $\Sigma^{\infty}S^0$. When the abelian group A is a ring, we obtain the ring structure on HA by linearly extending the product map $S^i \wedge S^j \simeq S^{i+j}$.

Let p be a prime, and let us specialize to the case $A = \mathbb{Z}/p = \mathbb{F}_p$, thought of as a ring. Henceforth, all spectra are understood to be p-localized. Then as above there is an equivalence of spectra $\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \simeq H\mathbb{F}_p$. The mod p infinite symmetric power model has a filtration by finite symmetric powers. Let $\mathrm{Sp}^{n}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \subset \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0)$ be the subspectrum

$$\mathrm{Sp}^{n}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) = \{a_1x_1 + \ldots + a_mx_m : a_i \in \{1, \ldots, p-1\}, x_i \in \Sigma^{\infty}S^0,\ \sum_{i=1}^{m} a_i \leq p^n\}.$$

The product restricts as $\mathrm{Sp}^{p^i}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \wedge \mathrm{Sp}^{p^j}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \to \mathrm{Sp}^{p^{i+j}}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0)$, and we consider the filtration of spectra

$$\Sigma^{\infty}S^0 = \mathrm{Sp}^{1}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \subset \mathrm{Sp}^{p}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \subset \mathrm{Sp}^{p^2}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \subset \cdots \subset \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \simeq H\mathbb{F}_p.$$

The above filtration was studied by Mitchell–Priddy, and ([18], Theorem A) identifies the n-th layer of the filtration as the n-th suspension of the Steinberg summand of the classifying space $B(\mathbb{Z}/p)^n$. That is, there is an equivalence of spectra

$$\Sigma^n e_n B(\mathbb{Z}/p)^n_+ \simeq \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^\infty S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^\infty S^0),$$

where e_n is the topological idempotent corresponding to the idempotent element e_n in the group ring $\mathbb{Z}_{(p)}[\mathrm{GL}_n(\mathbb{F}_p)]$. The aim of the present two-part paper is to prove an equivariant generalization of the theorem of Mitchell–Priddy.

Theorem 1. *Let G be a finite p-group. For each positive integer n, let $\mathrm{Sp}^n_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)$ be the n-th mod p symmetric power on the genuine G-spectrum $\Sigma^{\infty G} S^0$. Then there is an equivalence of genuine G-spectra*

$$\Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+ \simeq \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0).$$

The Steinberg summand construction e_n, and the G-equivariant classifying space construction $B_G\mathbb{Z}/p$ are both discussed in the companion paper, [20]. The G-equivariant classifying spaces $B_G\mathbb{Z}/p$ fit into a theory of equivariant principal bundles ([6]), but in our situation they can be built explicitly as lens spaces (See 46). The symmetric powers are topological in nature, while the Steinberg summand is algebraic, and Theorem 1 ties these two constructions together.

Note that by work of Lima-Filho and Dos Santos ([13], [4]), there is an equivalence equivalence of G-spectra $\mathrm{Sp}^\infty_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \simeq H\underline{\mathbb{F}}_p$. We also prove the following equivariant generalization of a well-known fact shown by Mitchell–Priddy.

Theorem 2. *Let G be any finite p group. The filtration $\{\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)\}_{n\geq 0}$ splits into its layers after smashing with $H\underline{\mathbb{F}}_p$. That is, there is an equivalence of $H\underline{\mathbb{F}}_p$-modules*

$$H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \simeq \bigvee_{n\geq 0} H\underline{\mathbb{F}}_p \wedge \Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+.$$

1.1 Intended application

Our intended application is towards a computation of the mod p equivariant dual Steenrod algebra $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$ when the group is $G = C_p$, the cyclic group of order p, and p is an odd prime. Theorem 1 identifies the layers in the symmetric power filtration of the genuine Eilenberg-Maclane G-spectrum $H\underline{\mathbb{F}}_p$. Theorem 2 tells us that after applying the functor $H\underline{\mathbb{F}}_p \wedge (-)$, the filtration splits. One may then construct an equivariant cellular filtration of the G-space $B_G\mathbb{Z}/p$, and use this filtration to compute the summands $H\underline{\mathbb{F}}_p \wedge \Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+$ in an invariant-theoretic manner inspired by Mitchell–Priddy ([18]). This computation will appear in a joint paper with Dylan Wilson.

For the prime $p = 2$, the C_2-equivariant dual Steenrod algebra has already been computed by Greenlees and later Hu–Kriz. Here, $\bigstar$ indicates a bigrading over the real representations of C_2, and σ denotes the sign representation. Hu–Kriz produced generators in $\pi^{C_2}_\bigstar(H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2)$

- $\xi_1, \xi_2, \ldots$ of degree $|\xi_i| = (2^i - 1)(1+\sigma)$, where ξ_i comes from the $2^i(1+\sigma)$-cell of the C_2-space $\mathbf{CP}^\infty \simeq B_{C_2}S^\sigma$.
- $\tau_0, \tau_1, \ldots$ of degree $|\tau_i| = 2^i + (2^i - 1)\sigma$, where τ_i comes from the $2^i(1+\sigma)$-cell of the C_2-space $B_{C_2}\mathbb{Z}/2$.

They then proved that

$$\pi_\star^{C_2}(H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2) \simeq (H\underline{\mathbb{F}}_2)_\bigstar[\xi_i, \tau_i]/\langle \tau_0 a_\sigma = u_\sigma + \eta_R(u_\sigma), \tau_i^2 = \tau_{i+1}a_\sigma + \xi_{i+1}\eta_R(u_\sigma)\rangle$$

An exposition of this computation is given in [14]. It is curious that this result bears a similarity to the classical odd primary dual Steenrod algebra computed by Milnor. Hu–Kriz ([8]) used the above computations to analyze the Adams-Novikov spectral sequence for $\mathrm{BP}_\mathbb{R}$, which converges to the 2-local stable homotopy groups of spheres.

1.2 Historical significance

The study of the mod p cohomology of the symmetric power filtration traces back to Nakaoka ([19]), who studied the symmetric power filtration for $H\mathbb{Z}$. The n-th layer in this filtration is given a name,

$$\mathrm{Sp}^{p^n}(\Sigma^\infty S^0)/\mathrm{Sp}^{p^{n-1}}(\Sigma^\infty S^0) =: \Sigma^n L(n).$$

It is classically known that $L(n) \simeq e_n(B(\mathbb{Z}/p)^n)^{\overline{\rho}_n}$, i.e. that $L(n)$ is the Steinberg summand of the Thom spectrum on $B(\mathbb{Z}/p)^n$ corresponding to the reduced regular representation $\overline{\rho}_n : (\mathbb{Z}/p)^n \to O(p^n - 1)$. See the paper of Arone-Dwyer-Lesh [3] for a proof of this fact. The closely related spectra $M(n) \simeq e_n B(\mathbb{Z}/p)^n_+$ which ([18]) appear as the layers in the mod p symmetric power were used by Mitchell provide a short proof of the Conner-Floyd conjecture ([17]). It is also a result of Welcher ([21]) that the spectrum $M(n)$ has chromatic type n.

Let $H^*(-)$ denote mod p cohomology. The mod p symmetric power filtration of $H\mathbb{F}_p$ closely reflects the structure of the Steenrod algebra. Mitchell–Priddy showed in [18] that $H^*(\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^\infty S^0))$ has a basis given by the classes $\theta^I(u_n)$ where $u_n \in H^0(\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^\infty S^0))$ is a generator, and I varies over admissible sequences of length at most n. Within this vector space, $H^*(\Sigma^n M(n))$ is the span of the classes $\theta^I(u_n)$ with length $\ell(I) = n$. A paper of Mitchell ([16]) demonstrates the algebra and invariant theory involved.

The Whitehead conjecture, proven by Kuhn ([12], [9]), states that the attaching maps yield a long exact sequence on homotopy groups

$$\cdots \longrightarrow L(3) \longrightarrow L(2) \longrightarrow L(1) \longrightarrow S^0 \longrightarrow H\mathbb{Z} \longrightarrow 0$$

This is a resolution of the spectrum $H\mathbb{Z}$ by spacelike spectra, i.e. spectra which are stable summands of spaces ([11]). Kuhn's orignial and also his modern proof ([10]) utilizes the fact that there are almost no $\mathcal{A}$-module maps between the $H^*(L(n))$'s, which is a kind of rigidity result. It is an observation due to Mitchell–Priddy that $M(n) \simeq L(n) \vee L(n-1)$, and there is a mod p version of the Whitehead conjecture. The main theorems of the present paper suggest that the above story may have an equivariant analogue.

1.3 Outline of the paper

Our proof of Theorem 1 is inspired by Mitchell–Priddy's proof of the nonequivariant analogue, which is by induction on n. The $n = 1$ case, namely that $\Sigma e_1(B\mathbb{Z}/p)_+ \simeq \Sigma(B\Sigma_p)_+ \simeq (H\mathbb{F}_p)_p/(H\mathbb{F}_p)_1$, is done by an explicit geometric argument. Now let $n \geq 2$. Both the Steinberg summands and the mod p symmetric powers are equipped with product maps. But the product maps on Steinberg summands have *retractions*. One has an intermediate inclusion of the Steinberg summand $e_nB(\mathbb{Z}/p)^n_+ \subset (e_1\boxtimes\cdots\boxtimes e_1)B(\mathbb{Z}/p)^n_+$. Now construct the commutative diagram

$$\begin{tikzcd}
\Sigma^n e_n B(\mathbb{Z}/p)^n_+ \arrow[r, hook, "\subset"'] \arrow[rr, dashed, bend left, "\mathrm{id}"] & \Sigma^n(e_1 B\mathbb{Z}/p_+)^{\wedge n} \arrow[r, "\text{product}"'] \arrow[d, equal] & \Sigma^n e_n B(\mathbb{Z}/p)^n_+ \\
 & (\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^\infty S^0)/\mathrm{Sp}^1_{\mathbb{Z}/p}(\Sigma^\infty S^0))^{\wedge n} \arrow[r, "\text{product}"'] & \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^\infty S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^\infty S^0)
\end{tikzcd} \quad . \tag{1.1}$$

Mitchell–Priddy prove that the zigzag composition $\Sigma^n e_n B(\mathbb{Z}/p)^n_+ \longrightarrow (H\mathbb{F}_p)_{p^n}/(H\mathbb{F}_p)_{p^{n-1}}$ from the top left to the bottom right of this diagram is an isomorphism on mod p cohomology, and therefore is a p-local equivalence. The proof is by induction on n and requires a particular lemma about the interaction of the e_n with the action of $\mathcal{A}$ in cohomology ([18], Section 5).

Our proof of Theorem 1 follows a similar strategy. The $n = 1$ case is proven in Section 5. For $n \geq 2$, we construct the analogous commutative diagram, and must prove that the zigzag composition $\Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+ \to \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)$ is a p-local equivalence of genuine G-spectra. By induction on the group G, it suffices to prove the zigzag composition is a mod p homology isomorphism on geometric fixed point spectra. Most of the hard supporting work in the proof is in computing the geometric fixed point spectra $\Phi^G(\mathbf{e}_n B_G(\mathbb{Z}/p)^n_+)$ (done in [20]) and $\Phi^G(\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)$, and computing the effect of the maps in the above diagram.

Here is an outline of this paper. In Section 2 we define the notion of the *primitives of a G-space X*, and prove a wedge sum decomposition of the geometric fixed points of the mod p symmetric powers of a suspension G-spectrum $\Sigma^{\infty G} X$ whose summands depend on the fixed point space X^G and the primitives of the G-space $S^{\infty \bar{\rho}_G}$. In Section 3, we prove that the primitives of $S^{\infty\bar{\rho}_G}$ are the homotopy orbit space of a certain subgroup complex. In Section 4, we compute the effect of the product maps relating the geometric fixed point spectra $\{\Phi^G \mathrm{Sp}^n_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)\}_{n\geq 0}$ as they pertain to our decomposition of the geometric fixed points.

In Section 5, we prove that there is an equivalence of G-spectra

$$\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^1_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \simeq \Sigma^{\infty G}(B_G\mathrm{Aff}_1)_+$$

where Aff_1 is the group of $p(p-1)$ affine permutations of the one-dimensional $\mathbb{F}_p$-line. This is the $n = 1$ case of Theorem 1. In Section 6, we combine the cumulative results of [20] and

Sections 2, 3, 4, to describe the zigzag composition of Diagram 1.1 on the homology of the geometric fixed points

$$f : H_*(\Phi^G(\Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+); \mathbb{F}_p) \to H_*(\Phi^G(\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)); \mathbb{F}_p).$$

We then use matrix algebra to prove that this map is an isomorphism, completing the proof of Theorem 1. Finally in Section 7 we prove Theorem 2 using a short and straightforward argument.

2 Fixed Points in Symmetric Powers

Let G be a finite group and let X be a pointed G-space. We decompose the G-fixed points of the infinite symmetric power $\mathrm{Sp}^\infty(X)$ by using a tower of fibrations. Specifically, we produce in Proposition 10 a finite sequence of topological abelian monoids

$$\mathrm{Sp}^\infty(X^G) = F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_m = \mathrm{Sp}^\infty(X)^G$$

and identify $F_{i-1} \to F_i$ as the homotopy fiber of a map from F_i to the infinite symmetric power of what we call a *primitive* of the G action on X, or $\mathrm{Pr}^{G/H}(X)$. There is a single primitive functor, and hence a single layer, for each conjugacy class of subgroups of G.

In Proposition 11, we generalize this decomposition to the free $\mathbb{Z}/p$–module generated by X, i.e. the mod–p symmetric powers, the case of Proposition 10 being $p = 0$. The main content of Proposition 11 is that this decomposition splits if G is a p-group.

In Proposition 19, we extend this decomposition to the case of the geometric fixed points Φ^G of the infinite symmetric power of a suspension G-spectrum $\Sigma^{\infty G} X$. In the context of spectra, we may exploit two further phenomena: fiber and cofiber sequences coincide, and the natural (nonequivariant) map

$$X \wedge \mathrm{Sp}^n(\Sigma^\infty S^0) \to \mathrm{Sp}^n(\Sigma^\infty X)$$

is an equivalence. This allows us to decompose the geometric fixed points of the finite symmetric power

$$\Phi^G \mathrm{Sp}^n(\Sigma^{\infty G} X)$$

in a similar manner to our decomposition for Sp^∞. This is Proposition 19. Combining Proposition 19 with Proposition 11, we obtain a decomposition of

$$\Phi^G \mathrm{Sp}^n_{\mathbb{Z}/p}(\Sigma^{\infty G} X)$$

when G is a p-group. This is Proposition 20. The infinite symmetric power of the genuine G-spectrum $\Sigma^{\infty G} S^0$ is a model for the equivariant Eilenberg-Maclane spectrum $H\underline{\mathbb{Z}}$, and therefore the finite symmetric powers are a filtration for this genuine G spectrum (Corollary 23). We thus deduce decompositions of the geometric fixed point spectra $\Phi^G(H\underline{\mathbb{Z}})$ and $\Phi^G(H\underline{\mathbb{F}}_p)$ in Propositions 24 and 25.

2.1 Symmetric Powers with Coefficients

Definition 3. Let X be a pointed G-space, and let $n \geq 1$ be a positive integer. The *n-th symmetric power* of X is the pointed G-space

$$\mathrm{Sp}^n(X) = X^{\times n}/\Sigma_n.$$

There are inclusions $\mathrm{Sp}^{n-1}(X) \hookrightarrow \mathrm{Sp}^n(X)$ given by adding a copy of the basepoint of X. The quotient of this inclusion is given by

$$\mathrm{Sp}^{n-1}(X) \to \mathrm{Sp}^n(X) \to X^{\wedge n}/\Sigma_n.$$

The *infinite symmetric power* of X is the colimit

$$\mathrm{Sp}^\infty X := \mathrm{colim}_{n\to\infty}\mathrm{Sp}^n(X) = \{(x_1 + x_2 + \ldots + x_n) : x_1, \ldots, x_n \in X\}.$$

The space $\mathrm{Sp}^\infty(X)$ is the free topological abelian monoid on X, and we write its elements as formal sums as above.

It is easily seen that $\mathrm{Sp}^n(-)$ is a functor from pointed G-spaces to pointed G-spaces enjoying the following properties.

1. There is an *addition* map given by formally summing,

$$\mathrm{Sp}^m(X) \times \mathrm{Sp}^n(X) \to \mathrm{Sp}^{m+n}(X).$$

2. Because the addition map is commutative, there is an *amalgamation* map,

$$\mathrm{Sp}^m(\mathrm{Sp}^n(X)) \to \mathrm{Sp}^{mn}(X).$$

3. There is a *multiplication* map given by formally multiplying,

$$\mathrm{Sp}^m(X) \wedge \mathrm{Sp}^n(Y) \to \mathrm{Sp}^{mn}(X \wedge Y).$$

More generally if M is any G-module, one may define the free topological M-module over X, denoted by $M \otimes X$ (see definition 2.1 in [4] for the precise definition).

$$M \otimes X = \{m_1x_1 + \ldots + m_nx_n : m_i \in M, x_i \in X\}$$

The group G acts on the space $M \otimes X$ by acting on both coordinates simultaneously. When M is the group $\mathbb{Z}$ with trivial G-action, then the inclusion $\mathrm{Sp}^\infty(X) \hookrightarrow \mathbb{Z} \otimes X$ is a weak equivalence as long as X^H is connected for every subgroup H.

When $M = \mathbb{Z}/p$ with trivial action, we given a special name to this functor.

Definition 4. For any pointed G-space X, define the *infinite mod p symmetric power* of X as the topological $\mathbb{Z}/p$-module,

$$\mathrm{Sp}^{\infty}_{\mathbb{Z}/p} := \mathbb{Z}/p \otimes X = \{(a_1x_1 + \ldots + a_nx_n : a_i \in \mathbb{Z}/p, x_i \in X\}.$$

There is an obvious surjection $\mathrm{Sp}^{\infty}(X) \to \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(X)$. We define the *$n$-th mod p symmetric power*, denoted $\mathrm{Sp}^{n}_{\mathbb{Z}/p}(X)$, as the image of $\mathrm{Sp}^{n}(X)$ under this surjection. Note that $\mathrm{Sp}^{n}(X) = \mathrm{Sp}^{n}_{\mathbb{Z}/p}(X)$ for $1 \leq n \leq p-1$.

It is easily seen that for any G-module A, the functor $A \otimes (-)$ takes cofiber sequences of pointed G-spaces to fiber sequences of unpointed G-spaces.

2.2 The Primitives of a G-space

Let G be a finite group, and let X be a G-space. For any two subgroups $H \subseteq K \subseteq G$, there is an inclusion of fixed point spaces $X^K \subseteq X^H$. Thus,

Definition 5. The fixed point spaces $\{X^H\}_{H \subseteq G}$ define a filtration of X indexed over the (opposite) poset of subgroups of G. We call this filtration the *fixed point filtration* of X.

The space X^H carries an action of the normalizer subgroup $N_G(H) = \{g \in G : gHg^{-1} = H\}$. Since H acts trivially on X^H, we obtain a residual action of the *Weyl group*, $W_G(H) = N_G(H)/H$ upon X^H.

Definition 6. We let $X(H)$ denote the H-th layer in the fixed point filtration, i.e.

$$X(H) := X^H / (\bigcup_{K \supsetneq H} X^K).$$

By definition, each point $x \in X(H)$ has isotropy group $\{g \in G : gx = x\} = H$. Therefore, the Weyl group $W_G(H)$ acts freely on the pointed space $X(H)$, and we define the quotient of this action as the *G/H-primitives of X*

$$\mathrm{Pr}^{G/H}(X) := X(H)/W_G(H).$$

Example 7. The first stage of the fixed point filtration is $\mathrm{Pr}^{G/G}(X) = X^G$.

For each subgroup H, $\mathrm{Pr}^{G/H}(-)$ is a functor from pointed G-spaces to pointed spaces enjoying the following properties.

1. Suppose that $H, H' \subseteq G$ are conjugate, that is there is some $g \in G$ such that $H' = gHg^{-1}$. Left action by g induces a homeomorphism

$$g : X(H) \to X(H').$$

and therefore, a homeomorphism $\mathrm{Pr}^{G/H}(X) \xrightarrow{\cong} \mathrm{Pr}^{G/H'}(X)$.

2. Let $[H]$ denote the set of subgroups of G which are conjugate to H. Then G acts on the wedge sum $\bigvee_{H' \in [H]} X(H')$, and the isotropy group of $x \in X(H')$ is H'. One may just as well define $\mathrm{Pr}^{G/H}(X)$ as the strict quotient $\mathrm{Pr}^{G/H}(X) = (\bigvee_{H' \in [H]} X(H'))/G$.

3. (Product) For $i = 1, 2$, let G_i be a finite group, $H_i \subseteq G_i$ a subgroup, and X_i a pointed G_i-space. Then there is a *product* map, which is a homeomorphism

$$\mathrm{Pr}^{G_1/H_1}(X_1) \wedge \mathrm{Pr}^{G_2/H_2}(X) \cong \mathrm{Pr}^{\frac{G_1 \times G_2}{H_1 \times H_2}}(X_1 \wedge X_2).$$

2.3 Subgroup Orderings

We would like to convert the fixed point filtration of X into a true filtration, and this motivates the following definition.

Definition 8. A *subgroup ordering for G* is a sequence of subgroups $H_0, H_1, \ldots, H_m \subseteq G$ satisfying two properties.

1. Every subgroup of G is conjugate to exactly one of the H_i.

2. If $i < j$, then no conjugate of H_i is a subgroup of H_j.

These two properties imply that $H_0 = G$ and $H_m = \{1\}$.

Given a subgroup ordering $H_0, \ldots, H_m$, one may construct an honest filtration of X,

$$X^G = X^{H_0} \subseteq \bigcup_{H' \in [H_1]} X^{H'} \subseteq \bigcup_{H' \in [H_2]} X^{H'} \subseteq \cdots \subseteq \bigcup_{H' \in [H_{m-1}]} X^{H'} \subseteq X^{H_m} = X. \tag{2.1}$$

It is clear that when G is a finite group, a subgroup ordering exists, and henceforth we will simply pick one. For our purposes, it does not matter which one.

2.4 Fixed Points and Primitives

Recall that $\mathrm{Sp}^\infty(X)$ denotes the free abelian monoid on the pointed G-space X. The G-fixed point space $\mathrm{Sp}^\infty(X)^G \subseteq \mathrm{Sp}^\infty(X)$ is the space of sums $(x_1 + \ldots + x_n)$ such that G permutes the points $x_1, \ldots, x_n$ in some fashion. In general, the topological abelian monoid $\mathrm{Sp}^\infty(X)^G$ is not a free one, i.e. there is no pointed space Y such that $\mathrm{Sp}^\infty(X)^G \cong \mathrm{Sp}^\infty(Y)$.

We will use the fixed point filtration of X to product a resolution of $\mathrm{Sp}^\infty(X)^G$ by the free abelian monoids $\mathrm{Sp}^\infty(\mathrm{Pr}^{G/H_i}(X))$ (Proposition 10). We will also show that the analogous resolution of $\mathrm{Sp}^\infty_{\mathbb{Z}/p}(X)^G$ splits as a Cartesian product when G is a p-group (Proposition 11). Our proofs will rely on the following lemma.

Lemma 9. *Let $H \subseteq G$ be a subgroup, and suppose that every point $x \in X$ has isotropy group conjugate to H. Let $[G : H] = |G|/|H|$ denote the index of the subgroup H. Then*

1. *The fixed point space $\mathrm{Sp}^\infty(X)^G$ is the free abelian monoid $\mathrm{Sp}^\infty(X)^G \cong \mathrm{Sp}^\infty(\mathrm{Sp}^{[G:H]}(X)^G)$, with the homeomorphism given by amalgamation.*

2. *There is a homeomorphism $\mathrm{Sp}^{[G:H]}(X)^G \cong \mathrm{Pr}^{G/H}(X)$.*

Proof. (1) is immediate from the observation that any sum $(x_1 + x_2 + \ldots + x_n) \in \mathrm{Sp}^\infty(X)^G$ decomposes as a sum of G-orbits isomorphic to G/H.

Now we prove (2). Let $a = [N_G(H) : H]$, and let $b = [G : N_G(H)]$, so that $[G : H] = ab$. By definition, $\mathrm{Pr}^{G/H}(X) = X^H/W_G(H)$. The transfer map

$$X^H/W_G(H) \to \mathrm{Sp}^{[G:H]}(X)^G$$

$$[x] \mapsto \sum_{g \in G/H} gx$$

has an inverse given as follows. If $(x_1 + \ldots + x_{[G:H]})$ is a G-fixed point of $\mathrm{Sp}^{[G:H]}(X)$, then there are a points x_j with isotropy group H — pick one. The image of x_j in the quotient space $X^H/W_G(H)$ does not depend on which point we pick, and

$$\sum_{g \in G/H} gx_j = (x_1 + \ldots + x_{[G:H]}).$$

□

Proposition 10. *Let $H_0, H_1, \ldots, H_m$ be a subgroup ordering (Definition 8) for G. Then the functor $(\mathrm{Sp}^\infty(-))^G$ has a filtration by functors valued in topological abelian monoids*

$$\mathrm{Sp}^\infty((-)^G) = F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_{m-1} \longrightarrow F_m = (\mathrm{Sp}^\infty(-))^G$$

where for each $1 \leq i \leq m$, the map $F_{i-1} \to F_i$ is the homotopy fiber of a fibration $F_i \to \mathrm{Sp}^\infty(\mathrm{Pr}^{G/H_i}(-))$.

Proof. Fix a pointed G-space X. For $0 \leq i \leq n$, let C_i denote the subspace $C_i = \bigcup_{H' \in [H_i]} X^{H'}$. Then as in Equation 2.1, one has the following filtration of X:

$$X^G = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{m-1} \subseteq C_m = X.$$

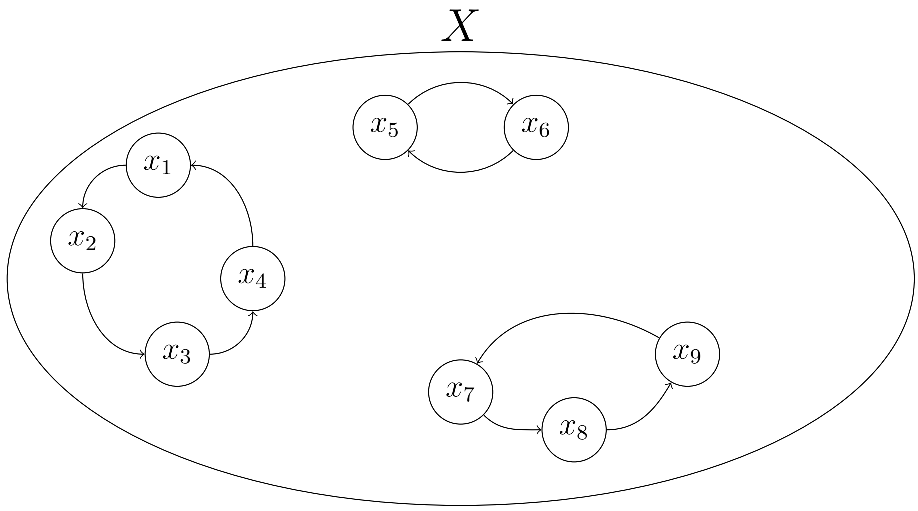
For each $0 \leq i \leq n$, let $F_i := \mathrm{Sp}^\infty(C_i)^G$. The functor $\mathrm{Sp}^\infty(-)$ (Definition 3) takes cofiber sequences to fiber sequences, and the functor $(-)^G$ preserves fiber sequences. Thus, for each $1 \leq i \leq n$, one has a fiber sequence

$$F_{i-1} \to F_i \to \mathrm{Sp}^\infty(C_i/C_{i-1})^G.$$

Every isotropy group of the cofiber C_i/C_{i-1} is conjugate to H_i. Thus, by Lemma 9, there is a homeomorphism

$$\mathrm{Sp}^\infty(C_i/C_{i-1})^G \cong \mathrm{Sp}^\infty(\mathrm{Pr}^{G/H_i}(C_i/C_{i-1})).$$

It is easily seen that $\mathrm{Pr}^{G/H_i}(C_i/C_{i-1}) = \mathrm{Pr}^{G/H_i}(X)$, by the definition of the primitive functor (6). Thus, the proof is complete. $\square$



Pictured: An element of $(\mathrm{Sp}^9(X))^G$ with three orbits.

Proposition 11. *Let G be a p-group. Let $H_0, H_1, \ldots, H_m$ be a subgroup ordering (8) for G. Then the functor $(\mathrm{Sp}^\infty_{\mathbb{Z}/p}(-))^G$ has a filtration by functors valued in topological abelian monoids*

$$\mathrm{Sp}^\infty_{\mathbb{Z}/p}((-)^G) = F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_{m-1} \longrightarrow F_m = (\mathrm{Sp}^\infty_{\mathbb{Z}/p}(-))^G$$

where for each $1 \le i \le m$, the map $F_{i-1} \to F_i$ is the homotopy fiber of a split fibration

$$\begin{tikzcd} F_{i-1} \arrow[r] & F_i \arrow[r] & \mathrm{Sp}^\infty_{\mathbb{Z}/p}(\mathrm{Pr}^{G/H_i}(-)) \arrow[l, dashed, bend right] \end{tikzcd}.$$

Thus, the functor $(\mathrm{Sp}^\infty_{\mathbb{Z}/p}(-))^G$ splits as a Cartesian product

$$(\mathrm{Sp}^\infty_{\mathbb{Z}/p}(-))^G \cong \prod_{i=0}^{m} \mathrm{Sp}^\infty_{\mathbb{Z}/p}(\mathrm{Pr}^{G/H_i}(-)).$$

Proof. Fix a pointed G-space X. For $0 \le i \le n$, let C_i denote the subspace $C_i = \bigcup_{H' \in [H_i]} X^{H'}$ as in Proposition 10. Let $F_i = (\mathrm{Sp}^\infty_{\mathbb{Z}/p}(C_i))^G$. Consider the continuous map of pointed spaces

$$f : X^{H_i} \to F_i$$

defined by $f(x) = \sum_{g \in G/H_i} gx$. Observe the following two properties of f.

- If the isotropy group of x is some $K \supsetneq H_i$, then the index $[K : H_i]$ is a power of p (because G is a p-group), and it follows that $f(x) = 0$. Therefore, the map f factors through the quotient $X^{H_i}/ \bigcup\limits_{K \supsetneq H_i} X^K$.
- It is easily checked that for any w in the Weyl group $W_G(H_i)$, we have $f(wx) = f(x)$.

So f descends to a map

$$\tilde{f} : \mathrm{Pr}^{G/H_i}(X) = (X^{H_i}/ \bigcup_{K \supsetneq H_i} X^K)/W \to (\mathbb{Z}/p \otimes \bigcup_{H \sim H_i} X^H)^G$$

Extend this map $\mathbb{Z}/p$-linearly to obtain the desired section map $(\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\mathrm{Pr}^{G/H_i}(X))) \to F_i$. It is easily seen that this map is the inverse to the fibration $F_i \to (\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\mathrm{Pr}^{G/H_i}(X)))$. $\square$

As a corollary, we deduce the existence of another structure map for the primitives functor when G is a finite p-group. Let $G' \subseteq G$ be any subgroup, and let $H \trianglelefteq G$ be a normal subgroup. Let $H' = G' \cap H$ denote the intersection. Then the G-set G/H decomposes as a G'-set into the disjoint union of $p^m = \frac{|G||H'|}{|G'||H|}$ copies of G'/H', i.e.

$$\mathrm{res}^G_{G'}(G/H) = (G'/H') \sqcup (G'/H') \sqcup \cdots \sqcup (G'/H').$$

The reason for assuming that $H \trianglelefteq G$ is to avoid subtlety about how $gHg^{-1} \cap G'$ may vary in size for different conjugate subgroups gHg^{-1}.

For any G-space X, consider the inclusion of fixed points

$$\iota : \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(X)^G \hookrightarrow \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(X)^{G'}.$$

Using the Cartesian product decomposition of Proposition 11, we obtain a $\mathbb{Z}/p$-module map $\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\mathrm{Pr}^{G/H}(X)) \to \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\mathrm{Pr}^{G'/H'}(X))$ which is the free $\mathbb{Z}/p$-module on a map

$$\mathrm{Pr}^{G/H}(X) \to \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\mathrm{Pr}^{G'/H'}(X)).$$

Definition 12. The map $\mathrm{Pr}^{G/H}(X) \to \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\mathrm{Pr}^{G'/H'}(X))$ described above is called the *restriction map* for primitives.

2.5 Passing to Spectra

For our purposes, a *spectrum* $\mathbf{X}$ is a sequence of pointed spaces $X_0, X_1, X_2, \ldots$ with specified structure maps $\Sigma X_n \to X_{n+1}$. A map between spectra $f : \mathbf{X} \to \mathbf{Y}$ is a collection of maps $f_n : X_n \to Y_n$ compatible with the structure maps. Such a map f is an *equivalence* if f_n is an equivalence on homotopy groups up to dimension $n + \ell(n)$ for every n, where $\ell : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is some function such that $\lim\limits_{n \to \infty} \ell(n) = \infty$. Intuitively, this means that as n grows, the connectivity of f_n grows at a rate faster than n.

When G is a finite or compact Lie group, there are two types of G-spectra to consider. See [7] for a detailed working guide to equivariant spectra.

Definition 13. A *naïve G-spectrum* $\mathbf{X}$ is a sequence of pointed G-spaces $X_0, X_1, X_2, \dots$ with specified G-equivariant maps $\Sigma X_n \to X_{n+1}$. For example, if X is a pointed G-space, then its naïve suspension spectrum is the naïve G-spectrum

$$\Sigma^\infty X = \{S^n \wedge X\}_{n \geq 0}.$$

For every subgroup $H \subseteq G$, one has the *fixed point spectrum* $\mathbf{X}^H = \{X_n^H\}_{n\geq 0}$. A map $f : \mathbf{X} \to \mathbf{Y}$ of naïve G-spectra is said to be an *equivalence* if $f^H : \mathbf{X}^H \to \mathbf{Y}^H$ is an equivalence of spectra for every subgroup $H \subseteq G$.

Definition 14. Let $RO(G)$ denote the orthogonal representation ring of G, and let ρ_G (resp. $\overline{\rho}_G$) denote the regular (resp. reduced regular) real representation of G. A *genuine G-spectrum* $\mathbf{X}$ is a collection of pointed G-spaces $\{X_V\}$ for every orthogonal G-representation V, with specified G-equivariant maps

$$S^{W-V} \wedge X_V \to X_W$$

whenever V is a subrepresentation of W. For example, if X is a pointed G-space, then its *genuine suspension spectrum* is the genuine G-spectrum

$$\Sigma^{\infty G} X = \{S^V \wedge X\}_{V \in RO(G)}.$$

The *geometric fixed point spectrum*, $\Phi^G \mathbf{X}$, is the spectrum whose n-th space is defined by

$$(\Phi^G \mathbf{X})_n = X^G_{n\rho_G}.$$

For each subgroup $H \subseteq G$, let $i_H^* \mathbf{X}$ denote the genuine H-spectrum given by restricting the action, and define $\Phi^H \mathbf{X} := \Phi^G i_H^* \mathbf{X}$. A map $f : \mathbf{X} \to \mathbf{Y}$ of genuine G-spectra is said to be an *equivalence* if $\Phi^H i_H^* f : \Phi^H i_H^* \mathbf{X} \to \Phi^H i_H^* \mathbf{Y}$ is an equivalence of spectra for every subgroup $H \subseteq G$. Note that of X is a pointed G-space, then $\Phi^G(\Sigma^{\infty G} X) = \Sigma^\infty(X^G)$.

Definition 15. Any naïve G-spectrum $\mathbf{X} = \{X_n\}_{n\geq 0}$ may be promoted to a genuine G-spectrum $\{X_V\}_{V \in RO(G)}$ as follows. For any orthogonal representation V, let V^G denote the fixed points and $\overline{V} \subset V$ denote an orthogonal complement to V^G. Note that $\overline{V}^G = 0$. Then we define pointed G-spaces X_V indexed over the orthogonal representations V by

$$X_V := S^{\overline{V}} \wedge X_{V^G}.$$

This construction defines a functor i_* from naïve G-spectra to genuine G-spectra, and this functor can be viewed as inverting the representation sphere $S^{\overline{\rho}_G}$. It is easily checked that

$$\Phi^G(i_* \mathbf{X}) \simeq \mathbf{X}^G.$$

Let n be a positive integer. The n-th symmetric power functor $\mathrm{Sp}^n(-)$ may be applied to a naïve or genuine G-spectrum by application to each component space. We decompose the geometric fixed points of the n-th symmetric power of a genuine suspension spectrum of a pointed G-space X via a filtration of spectra

$$\mathrm{Sp}^n(\Sigma^\infty X^G) = F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_m = \Phi^G(\mathrm{Sp}^n(\Sigma^{\infty G} X))$$

where the cofiber of the map $F_{i-1} \to F_i$ is given by a smash product of three factors

$$\mathrm{cof}(F_{i-1} \to F_i) \simeq X^G \wedge \mathrm{Sp}^{\lfloor n/c_i \rfloor}(\Sigma^\infty S^0) \wedge \mathrm{Pr}^{G/H_i}(S^{\infty\overline{\rho}_G})$$

where c_i is the index of the subgroup H_i in G. This is Proposition 19. Our proof relies on Lemmas 16, 17, and 18 about finite symmetric powers in the stable setting, stated below and proven at the end of this section. These are equivariant analogues to several lemmas in ([2], Section 7).

Lemma 16. *The natural transformation*

$$(-) \wedge \mathrm{Sp}^n(\Sigma^{\infty G} S^0) \to \mathrm{Sp}^n(\Sigma^{\infty G}(-))$$

is an equivalence of functors from pointed G-spaces to genuine G-spectra. In particular, this functor preserves cofiber sequences.

Lemma 17. *The functor* $\mathrm{Sp}^n(\Sigma^\infty(-))$ *from pointed G-spaces to naïve G-spectra preserves cofiber sequences.*

Note: It is NOT true that $X \wedge \mathrm{Sp}^n(\Sigma^\infty S^0) \to \mathrm{Sp}^n(\Sigma^\infty X)$ is an equivalence! For example, if X is a G-space whose fixed points are contractible, then the two naïve G-spectra above have different G-fixed points.

Lemma 18. *For any nonnegative integer ℓ, there is an inclusion of G-representation spheres $S^{\ell\rho_G} \hookrightarrow S^\ell \wedge S^{\infty\overline{\rho}_G}$. The resulting natural transformation of functors from pointed G-spaces to spectra is an equivalence.*

$$\Phi^G(\mathrm{Sp}^n(\Sigma^{\infty G}(-))) \to \mathrm{Sp}^n(\Sigma^\infty(S^{\infty\overline{\rho}_G} \wedge (-)))^G$$

Given these three lemmas, we state and prove the main proposition of this section.

Proposition 19. *Let $H_0, \ldots, H_m$ be a subgroup ordering of G, and for each i let $c_i = [G : H_i]$. Then the functor $\Phi^G(\mathrm{Sp}^n(\Sigma^{\infty G}(-)))$ from pointed G-spaces to spectra has a filtration by functors*

$$\mathrm{Sp}^n(\Sigma^\infty(-)^G) = F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_m = \Phi^G(\mathrm{Sp}^n(\Sigma^{\infty G}(-)))$$

where $F_{i-1} \to F_i$ is the homotopy fiber of a fibration

$$F_i \to (-)^G \wedge \mathrm{Sp}^{\lfloor n/c_i \rfloor}(\Sigma^\infty S^0) \wedge \mathrm{Pr}^{G/H_i}(S^{\infty\overline{\rho}_G}).$$

Proof. Let X be any pointed G-space. By Lemma 16, the natural inclusion

$$X^G \wedge \Phi^G(\mathrm{Sp}^n(\Sigma^{\infty G} S^0)) \to \Phi^G(\mathrm{Sp}^n(\Sigma^{\infty G} X)$$

is an equivalence of spectra. It therefore suffices to prove the proposition when $X = S^0$. By Lemma 18,

$$\Phi^G(\mathrm{Sp}^n(\Sigma^{\infty G} S^0)) \simeq \mathrm{Sp}^n(\Sigma^\infty(S^{\infty\overline{\rho}_G}))^G$$

so we instead compute $\mathrm{Sp}^n(\Sigma^\infty(S^{\infty\overline{\rho}_G}))^G$. Consider the isotropy filtration of $S^{\infty\overline{\rho}_G}$.

$$S^0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_m \longrightarrow S^{\infty\overline{\rho}_G} \qquad ; \qquad C_i = \bigcup_{H' \in [H_i]} S^{\infty\overline{\rho}_G^{H'}}$$

where the cofiber of the inclusion $C_{i-1} \to C_i$ is denoted $S^{\infty\overline{\rho}_G}(H_i)$. Applying the functor $\mathrm{Sp}^n(\Sigma^\infty(-))^G$ to this diagram, one obtains a filtration

$$\mathrm{Sp}^n(\Sigma^\infty S^0) \longrightarrow \mathrm{Sp}^n(\Sigma^\infty C_1)^G \longrightarrow \cdots \longrightarrow \mathrm{Sp}^n(\Sigma^\infty C_m)^G \longrightarrow \mathrm{Sp}^n(\Sigma^\infty S^{\infty\overline{\rho}_G})^G$$

By Proposition 17, the cofiber of the inclusion $\mathrm{Sp}^n(\Sigma^\infty C_{i-1})^G \to \mathrm{Sp}^n(\Sigma^\infty C_i)^G$ is

$$\mathrm{Sp}^n(\Sigma^\infty S^{\infty\overline{\rho}_G}(H_i))^G \simeq \mathrm{Sp}^{\lfloor n/c_i \rfloor}(\Sigma^\infty \mathrm{Pr}^{G/H_i}(S^{\infty\overline{\rho}_G}))$$

which completes the proof. $\square$

Since the associated fibration sequences in the mod p symmetric powers split (via Proposition 11), we deduce the following result in a manner similar to the proof of Proposition 19 above.

Proposition 20. *Let $H_0, \dots, H_m$ be a subgroup ordering of G, and for each i let $c_i = [G : H_i]$. There is an equivalence between the two functors from pointed G-spaces to nonequivariant spectra,*

$$\Phi^G \mathrm{Sp}^n_{\mathbb{Z}/p}(\Sigma^{\infty G}(-)) \simeq (-)^G \wedge \bigvee_{i=0}^{m} (\mathrm{Sp}^{\lfloor n/c_i \rfloor}_{\mathbb{Z}/p}(\Sigma^\infty S^0) \wedge \mathrm{Pr}^{G/H_i}(S^{\infty\overline{\rho}_G})).$$

We now prove Lemmas 16, 17, and 18.

Lemma 21. *Let $\overline{\mathbb{R}^n}$ denote the reduced standard $(n-1)$-dimensional representation of Σ_n. Let Γ be a subgroup of the Cartesian product $G \times \Sigma_n$ such that $\Gamma \cap (1 \times \Sigma_n)$ is nontransitive. Then the Γ-fixed points of the representation $\rho_G \otimes \overline{\mathbb{R}^n}$ are nonzero.*

Proof. Let $\Gamma' = \Gamma \cap (1 \times \Sigma_n)$. Since Γ' is nontransitive, there is a vector $w \in \overline{\mathbb{R}^n}$ which is fixed under Γ'. Pick any vector $v \in \rho_G$ such that the images $\{gv\}_{g \in G}$ are linearly indepdent, and consider the vector

$$u := \sum_{\gamma \in \Gamma} \gamma(v \otimes w)$$

Clearly u is fixed by Γ, so it suffices to prove that u is nonzero. Let H denote the projection of Γ onto G, and let Ψ denote the projection of Γ onto Σ_n. It is easily checked that Γ' is a normal subgroup of Ψ, and so Γ may be thought of as the graph of a homomorphism $f : H \to \Psi/\Gamma'$. Then

$$u = \sum_{\gamma\in\Gamma} \gamma(v \otimes w) = \sum_{h\in H} hv \otimes (|\Gamma'| \cdot f(h)w)$$

which is clearly nonzero because the images $\{hv\}_{h\in H}$ are linearly independent vectors in the representation ρ_G. □

Proof of Lemma 16. We use induction on n. The case $n = 1$ is a tautology. Now consider general n. We will show that for any positive integer ℓ and any subgroup $H \subset G$, the natural map

$$(X \wedge \mathrm{Sp}^n(S^{\ell\rho_G}))^H \to (\mathrm{Sp}^n(X \wedge S^{\ell\rho_G}))^H$$

is an isomorphism on homotopy groups up through dimension $(2\ell - 1)$. Note that for any pointed G-space Y, $\mathrm{Sp}^n(Y)/\mathrm{Sp}^{n-1}(Y) \simeq Y^{\wedge n}/\Sigma_n$, and so it suffices for us to show that the natural map

$$X \wedge (S^{\ell\rho_G})^{\wedge n}/\Sigma_n \to (X \wedge S^{\ell\rho_G})^{\wedge n}/\Sigma_n$$

is $(2\ell-1)$-connected. To prove this, it suffices to show that for any subgroup $\Gamma \subset G \times \Sigma_n$, the map of fixed point spaces

$$(X \wedge (S^{\ell\rho_G})^{\wedge n})^\Gamma \to (X^{\wedge n} \wedge (S^{\ell\rho_G})^{\wedge n})^\Gamma$$

is $(2\ell - 1)$-connected. Let H denote the projection of Γ onto G. The two terms above can be rewritten as follows.

$$X^H \wedge (S^{\ell\rho_G})^H \wedge (S^{\ell(\rho_G\otimes\overline{\mathbb{R}^n})})^\Gamma \to (X^{\wedge n})^\Gamma \wedge (S^{\ell\rho_G})^H \wedge (S^{\ell(\rho_G\otimes\overline{\mathbb{R}^n})})^\Gamma$$

If $\Gamma \cap (1 \times \Sigma_n)$ is transitive, then $(X^{\wedge n})^\Gamma = X^H$ and the map above is the identity map, so we are done. Suppose instead that $\Gamma \cap (1 \times \Sigma_n)$ is nontransitive. Then $(S^{\ell\rho_G})^H$ has dimension at least i, and by Lemma 21, $(S^{\ell(\rho_G\otimes\overline{\mathbb{R}^n})})^\Gamma$ has dimension at least i. Thus, the two spaces above are each at least a 2ℓ-fold suspension, and therefore the map shown is automatically $(2\ell - 1)$-connected. □

Proof of Lemma 17. We use induction on n. The base case $n = 1$ is a tautology. Now consider general n. Let $X \to Y \to Z$ be a cofiber sequence of pointed G-spaces. By an argument analogous to the previous lemma, it suffices to show that for any subgroup $\Gamma \subset G \times \Sigma_n$, the map of fixed point spaces

$$((S^\ell)^{\wedge n} \wedge Y^{\wedge n}/X^{\wedge n})^\Gamma \to ((S^\ell)^{\wedge n} \wedge Z^{\wedge n})^\Gamma$$

is $(2\ell - 1)$-connected. Define Ψ to be the projection of Γ onto Σ_n. If Ψ is nontransitive, then $(S^{\ell n})^\Gamma = S^{\ell\cdot o(\Psi)}$ where $o(\Psi)$ is the number of orbits of $\{1, \ldots, n\}$ under the action of Ψ. Then the two spaces above are each at least a 2ℓ-fold suspension, and the map above is automatically $(2\ell-1)$-connected.

Suppose that Ψ is transitive. Let $H = \Gamma \cap (G \times 1)$. Then for any G-space A, the map

$$f : (A^{\wedge n})^{\Gamma} \to A^{H}, \qquad f(a_1, \ldots, a_n) = a_1$$

is a homeomorphism. This is true because a_1 can be freely chosen to be any H-fixed point of A, and each a_i is uniquely determined by the point a_1. Therefore, $(Y^{\wedge n}/X^{\wedge n})^{\Gamma} \simeq Y^H/X^H$ and $(Z^{\wedge n})^{\Gamma} \simeq Z^H$. The pointed spaces Y^H/X^H and Z^H are homotopy equivalent, and therefore the map of Γ-fixed points is an equivalence. □

Proof of Lemma 18. By induction on n, it suffices to prove that the map of pointed spaces

$$((S^{\ell} \wedge S^{\ell\overline{\rho}_G})^{\wedge n})^{\Gamma} \to ((S^{\ell} \wedge S^{\infty\overline{\rho}_G})^{\wedge n})^{\Gamma}$$

is a $(2\ell-1)$-equivalence. If the Γ-fixed points of the representation $(\overline{\rho}_G \otimes \mathbb{R}^n)$ are zero, then the above map is an equivalence. If the Γ-fixed points of the representation $(\overline{\rho}_G \otimes \mathbb{R}^n)$ are nonzero, then both of the spaces above are 2ℓ-fold suspensions, and the map is a $(2\ell-1)$-equivalence. □

2.6 Equivariant Eilenberg-Maclane Spectra

Let X be a based G-CW complex and let M be a G-module. Recall, either from Section 2.1 or from ([4], Definition 2.1) that $M \otimes X$ denotes the free topological M-module over X where G acts diagonally. When $M = \mathbb{Z}$, one obtains a G-space weakly equivalent to the infinite symmetric power of X, and when $M = \mathbb{Z}/p$, one obtains the infinite mod p symmetric power of X. These topological modules were studied in the papers [13] and [4], where it was shown that the equivariant homotopy groups of $M \otimes X$ encode the equivariant homology groups of X.

Definition 22. Let G be a finite group, and let $\underline{M}$ be a Mackey functor for the group G. Then there is a genuine G-spectrum $H\underline{M}$, called the *Eilenberg-Maclane spectrum* for $\underline{M}$, which represents the functor for Bredon (co)homology with coefficients in $\underline{M}$.

We will not use equivariant homotopy groups, equivariant homology groups, or Mackey functors in this paper, and so we will not define them — the reader may consult [5] for a reference. However, our main result is motivated by a desire to decompose the equivariant Eilenberg-Maclane spectrum $H\underline{\mathbb{F}}_p$, which is the genuine G-spectrum representing Bredon cohomology with coefficients in the trivial G-module $\mathbb{F}_p$. Thus, we illuminate the connection to mod p symmetric powers.

The primary result we are interested in is ([4], Theorem 1.1), which states that $M \otimes X$ is an equivariant infinite loop space and there is an equivalence natural in both the G-space X and the G-module M,

$$\pi_V^G(M \otimes X) \cong \tilde{H}_V^G(X; \underline{M}).$$

When G is the trivial group, the above result reduces to the classical Dold-Thom theorem. If we let X denote the representation sphere S^V, then one obtains an equivalence of equivariant infinite loop spaces

$$M \otimes S^V \simeq K(\underline{M}, V).$$

Using naturality, we obtain an equivalence of genuine G-spectra

$$M \otimes \Sigma^{\infty G} S^0 \simeq H\underline{M}.$$

If M is a ring with a unit $1 \in M$, then $H\underline{M}$ is a ring spectrum, and the inclusion $\Sigma^{\infty G} S^0 \hookrightarrow M \otimes \Sigma^{\infty G} S^0 \simeq H\underline{M}$ is the unit of the genuine G-spectrum $H\underline{M}$. The following corollary is immediate.

Corollary 23. *There are filtrations of genuine G-spectra*

$$\Sigma^{\infty G} S^0 \simeq \mathrm{Sp}^1(\Sigma^{\infty G} S^0) \subseteq \mathrm{Sp}^2(\Sigma^{\infty G} S^0) \subseteq \mathrm{Sp}^3(\Sigma^{\infty G} S^0) \subseteq \cdots \subseteq \mathrm{Sp}^\infty(\Sigma^{\infty G} S^0) \simeq H\underline{\mathbb{Z}}$$

$$\Sigma^{\infty G} S^0 \simeq \mathrm{Sp}^1_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \subseteq \mathrm{Sp}^2_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \subseteq \mathrm{Sp}^3_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \subseteq \cdots \subseteq \mathrm{Sp}^\infty_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \simeq H\underline{\mathbb{F}}_p.$$

The multiplication map on finite symmetric powers (resp. finite mod p symmetric powers) in Definition 3 filters the ring structure of $H\underline{\mathbb{Z}}$ (resp. $H\underline{\mathbb{F}}_p$),

$$\mathrm{Sp}^m(\Sigma^{\infty G} S^0) \wedge \mathrm{Sp}^n(\Sigma^{\infty G} S^0) \to \mathrm{Sp}^{mn}(\Sigma^{\infty G} S^0)$$

$$\mathrm{Sp}^m_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge \mathrm{Sp}^n_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \to \mathrm{Sp}^{mn}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0).$$

Thus, our work in this section yields computations of the geometric fixed point spectra $\Phi^G(H\underline{\mathbb{Z}})$ and $\Phi^G(H\underline{\mathbb{F}}_p)$.

Proposition 24. *Let $H_0, \ldots, H_m$ be a subgroup ordering of G. The geometric fixed point spectrum $\Phi^G(H\underline{\mathbb{Z}})$ has a filtration by $H\mathbb{Z}$-modules,*

$$H\mathbb{Z} = F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_m = \Phi^G(H\underline{\mathbb{Z}})$$

where $F_{i-1} \to F_i$ is the homotopy fiber of a fibration of $H\mathbb{Z}$-modules

$$F_i \to H\mathbb{Z} \wedge \mathrm{Pr}^{G/H_i}(S^{\infty \bar{\rho}_G}).$$

Proof. Apply Proposition 19. □

Proposition 25. *Let $H_0, \ldots, H_m$ be a subgroup ordering of G. There is an equivalence of $H\mathbb{F}_p$-modules*

$$\Phi^G(H\underline{\mathbb{F}}_p) \simeq \bigvee_{i=0}^{m} (H\mathbb{F}_p \wedge \mathrm{Pr}^{G/H_i}(S^{\infty \bar{\rho}_G})).$$

Proof. Apply Proposition 20. □

In the next section, we compute an explicit expression for the space of primitives $\mathrm{Pr}^{G/H}(S^{\infty \bar{\rho}_G})$.

3 The Primitives of an infinite representation sphere

Let G be a finite group, and let X be a pointed G-space. We compute the primitives of the G-equivariant suspension spectrum of X, which is an equivalent task to describing the space $\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G} \wedge X)$ (Proposition 32). To give the explicit description, we must give two definitions.

Definition 26. For any unpointed space Y, we use the notation $Y^\Diamond$ to denote the unreduced suspension of Y. The space $Y^\Diamond$ will be considered based, with the image of $Y \times \{0\} \hookrightarrow Y^\Diamond$ as the basepoint. In this way, even if Y is not a based space, $Y^\Diamond$ is.

Definition 27. For any subgroup $H \subseteq G$, we write $\mathcal{P}(G)_{\supset H}$ to be the poset of proper subgroups of G which strictly contain H.

Let $N_G(H) := \{g \in G : gHg^{-1} = H\}$ denote the normalizer subgroup of H, and let $W_G(H) := N_G(H)/H$ denote the Weyl group. Then Proposition 32 states that

$$\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G} \wedge X) \simeq \Sigma(\mathcal{P}(G)^\Diamond_{\supset H})_{hW_G(H)} \wedge X^G.$$

The main idea of the proof is that one may cover $\bigcup\limits_{K \supsetneq H} S^{\infty\overline{\rho}^K_G}$ by the contractible subspaces $S^{\infty\overline{\rho}^K_G}$ where K varies over the poset $\mathcal{P}(G)_{\supset H} \cup \{G\}$. With this covering one may prove, using Lemma 30, that $\bigcup\limits_{K \supsetneq H} S^{\infty\overline{\rho}^K_G} \simeq \mathcal{P}(G)^\Diamond_{\supset H}$.

When we restrict our attention to p-groups G, a further simplification occurs. Any p-group G has a maximal elementary abelian quotient $G \to G/F$, where F denotes the kernel of the quotient map (Definition 33). If H does contain the subgroup F, then we show the poset $\hat{\mathcal{P}}(G)_{\supset H}$ is contractible (Lemma 34). This allows us to simplify the expression above, and this simplification is stated in Proposition 35. Proposition 29 combines all of the results of this section into a form we will use later.

Definition 28. Let G be a finite p-group. We let $\mathcal{C}$ be the set

$$\mathcal{C} := \{H \trianglelefteq G : G/H \text{ is an elementary abelian } p\text{-group}\}.$$

The set $\mathcal{C}$ is closed under taking intersections, and its minimal element F is the Frattini subgroup of G (Definition 33). A subgroup H is contained in $\mathcal{C}$ if and only if $F \subseteq H$.

Proposition 29. *Let G be a p-group. Then there is an equivalence of $\mathbb{Z}/p$-modules,*

$$\mathrm{Sp}^\infty_{\mathbb{Z}/p}(S^{\infty\overline{\rho}_G})^G \simeq \prod_{H \in \mathcal{C}} \mathrm{Sp}^\infty_{\mathbb{Z}/p}(\Sigma\mathcal{P}(G)^\Diamond_{\supset H} \wedge B(G/H)_+).$$

Proof. Proposition 11 implies that there is a homeomorphism

$$\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(S^{\infty\bar{\rho}_G})^G \cong \prod_{i=0}^{m} \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\mathrm{Pr}^{G/H_i}(S^{\infty\bar{\rho}_G})),$$

where $H_0, \ldots, H_m$ are representatives for the conjugacy classes of subgroups of G. Proposition 35 implies that

$$\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G}) \simeq \begin{cases} \Sigma\mathcal{P}(G)^{\diamondsuit}_{\supset H} \wedge B(G/H)_+ & H \in \mathcal{C} \\ \star & H \notin \mathcal{C} \end{cases}.$$

The result immediately follows. □

3.1 Coverings and Posets

We familiarize the reader with a computational tool which will be crucial to us. Let $\mathcal{P}$ be a finite poset. Suppose that Y is a pointed topological space. A functor $\mathbf{Y} : \mathcal{P} \to \mathrm{Top}_*$ is called a *covering* of Y if $\mathrm{hocolim}_{\mathcal{P}}\mathbf{Y} \simeq Y$. If $\mathbf{Y}$ and $\mathbf{Y}'$ are two functors such that there is a natural homotopy equivalence between them, then $\mathrm{hocolim}_{\mathcal{P}}\mathbf{Y} \simeq \mathrm{hocolim}_{\mathcal{P}}\mathbf{Y}'$. If we replace $\mathbf{Y}$ by a diagram whose maps are all cofibrations, then the homotopy colimit is the union $\bigcup_{a\in\mathcal{P}} \mathbf{Y}(a)$.

Now suppose that for any two elements $a, b \in \mathcal{P}$, there is a unique maximal element $a \wedge b$ such that $a \geq a \wedge b$ and $b \geq a \wedge b$. Such posets are called *meet semilattices.* If $\mathcal{P}$ is a finite meet semilattice, then $\mathcal{P}$ has a unique minimal element, which we denote by 0. Let $\mathcal{P}_{>0} = \mathcal{P} - \{0\}$.

Lemma 30. *Consider a functor* $\mathbf{Y} : \mathcal{P} \to \mathrm{Top}_*$ *with the property that* $\mathbf{Y}(a) \simeq \star$ *whenever* $a \neq 0$. *Then* $\mathrm{hocolim}_{\mathcal{P}}\mathbf{Y} \simeq \mathbf{Y}(0) \wedge \mathcal{P}^{\diamondsuit}_{>0}$.

Proof. For any $a \in \mathcal{P}$, let $\mathcal{P}_{\backslash a} = \{x \in \mathcal{P} : x \leq a\}$. The pointed space $\mathbf{Y}(0) \wedge \hat{\mathcal{P}}^{\diamondsuit}$ has a covering $\mathbf{Y}'$ defined by

$$\mathbf{Y}'(a) = \mathbf{Y}(0) \wedge (\mathcal{P}_{\backslash a} - \{0\})$$

If $a \neq 0$, then the poset $\mathcal{P}_{\backslash a} - \{0\}$ has a maximal element (namely, a), so it is contractible and thus $\mathbf{Y}'(a) \simeq \star$. If $a = 0$, it's obvious that $\mathcal{P}_{\backslash 0} - \{0\}$ is empty, and therefore its unreduced suspension is S^0. Thus, $\mathbf{Y}' \simeq \mathbf{Y}$, and so

$$\mathrm{hocolim}_{\mathcal{P}}\mathbf{Y} \simeq \mathrm{hocolim}_{\mathcal{P}}\mathbf{Y}' \simeq \mathbf{Y}(0) \wedge (\mathcal{P}_{>0})^{\diamondsuit}$$

□

Now suppose that $\mathcal{P}$ and $\mathcal{Q}$ are two posets and $f : \mathcal{P} \to \mathcal{Q}$ is an order-preserving map. Let us suppose that $\mathbf{X} : \mathcal{P} \to \mathrm{Top}_*$ and $\mathbf{Y} : \mathcal{Q} \to \mathrm{Top}_*$ are two functors such that

1. For every $a \in \mathcal{P}$, there is a map $g_a : \mathbf{X}(a) \to \mathbf{Y}(f(a))$.

2. If $a \leq b$, then the following diagram commutes

$$\begin{array}{ccc} \mathbf{X}(a) & \xrightarrow{g_a} & \mathbf{Y}(f(a)) \\ \downarrow & & \downarrow \\ \mathbf{X}(b) & \xrightarrow{g_b} & \mathbf{Y}(f(b)) \end{array}$$

Then there is an induced map $\mathrm{hocolim}_{\mathcal{P}}\mathbf{X} \to \mathrm{hocolim}_{\mathcal{Q}}\mathbf{Y}$. The lemma below states that the homotopy equivalence of Lemma 30 is functorial in a strong sense.

Lemma 31. *Suppose further that $\mathcal{P}$ and $\mathcal{Q}$ are finite meet semilattices and for every $a, b \in \mathcal{P}$, $f(a \wedge b) = f(a) \wedge f(b)$. If $\mathbf{X}(a) \simeq \star$ whenever $a \neq 0$ and $\mathbf{Y}(c) \simeq \star$ whenever $c \neq 0$, then the map $\mathrm{hocolim}_{\mathcal{P}}\mathbf{X} \to \mathrm{hocolim}_{\mathcal{Q}}\mathbf{Y}$ is the map*

$$g_0 \wedge f : \mathbf{X}(0) \wedge (\mathcal{P}_{>0})^{\Diamond} \to \mathbf{Y}(0) \wedge (\mathcal{Q}_{>0})^{\Diamond}$$

Proof. As in the proof of the previous lemma, replace $\mathbf{X}$ with the homotopy equivalent functor $\mathbf{X}'$ defined by $\mathbf{X}'(a) = \mathbf{X}(0) \wedge (\mathcal{P}_{\backslash a} - \{0\})^{\Diamond}$, and replace $\mathbf{Y}$ with the homotopy equivalent functor $\mathbf{Y}'$ defined by $\mathbf{Y}'(c) = \mathbf{Y}(0) \wedge (\mathcal{Q}_{\backslash c} - \{0\})^{\Diamond}$. □

3.2 The Primitives of an infinite representation sphere

Proposition 32. *Let G be a finite group, and let X be a pointed G-space. Then for any subgroup $H \subset G$,*

$$\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G} \wedge X) \simeq \Sigma(\mathcal{P}(G)^{\Diamond}_{\supset H})_{hW} \wedge X^G$$

where $W = W_G(H)$ is the Weyl group of H.

Proof. If $H = G$, then $\mathrm{Pr}^{G/G}(S^{\infty\bar{\rho}_G} \wedge X) = S^0 \wedge X^G$, and by convention $\Sigma(\mathcal{P}(G)^{\Diamond}_{\supset G}) \wedge X^G \simeq \Sigma(S^{-1}) \wedge X^G$. This deals with the case $H = G$, so let us henceforth assume $H \neq G$.

By definition, $\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G} \wedge X)$ sits in a cofiber sequence

$$W\backslash(\bigcup_{K \supsetneq H} S^{\infty\bar{\rho}^K_G} \wedge X^K) \longrightarrow W\backslash(S^{\infty\bar{\rho}^H_G} \wedge X^H) \longrightarrow \mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G} \wedge X)$$

Every point in the the quotient space $(S^{\infty\bar{\rho}^H_G} \wedge X^H)/(\bigcup_{K \supsetneq H} S^{\infty\bar{\rho}^K_G} \wedge X^K)$ has isotropy group equal to H, and therefore W acts freely on this quotient space. So we may replace the strict quotient by the homotopy quotient.

$$(\bigcup_{K \supsetneq H} S^{\infty\bar{\rho}^K_G} \wedge X^K)_{hW} \longrightarrow (S^{\infty\bar{\rho}^H_G} \wedge X^H)_{hW} \longrightarrow \mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G} \wedge X)$$

Since $H \subsetneq G$, the W-space $S^{\infty\bar{\rho}^H_G} \wedge X^H \wedge EW_+$ has underlying points $S^\infty \wedge X^H \wedge EW_+ \simeq \star$, and its fixed points under any nonzero subgroup of W are also contractible because W

acts freely on EW_+. Therefore, $S^{\infty\overline{\rho}_G^H} \wedge X^H \wedge EW_+$ is contractible as a W-space, and so $(S^{\infty\overline{\rho}_G^H} \wedge X^H)_{hW} \simeq \star$. It follows that $\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G} \wedge X) \simeq \Sigma(\bigcup_{K \supsetneq H} S^{\infty\overline{\rho}_G^K} \wedge X^K)_{hW}$. Let $\hat{\mathcal{P}}(G)_{\supset H}$ denote the poset of all subgroups strictly containing H. The space $\bigcup_{K \supsetneq H} S^{\infty\overline{\rho}_G^K} \wedge X^K$ has a covering $\mathbf{Y}(-)$ indexed over $\hat{\mathcal{P}}(G)_{\supset H}$. It is defined by

$$\mathbf{Y}(K) = S^{\infty\overline{\rho}_G^K} \wedge X^K \simeq \begin{cases} \star & K \neq G \\ X^G & K = G \end{cases}$$

Therefore, Lemma 30 implies that

$$\bigcup_{K \supsetneq H} S^{\infty\overline{\rho}_G^K} \wedge X^K = \mathrm{hocolim}_{\hat{\mathcal{P}}_{\supset H}} \mathbf{Y} \simeq \mathbf{Y}(G) \wedge \mathcal{P}^{\lozenge}_{\supset H} \simeq X^G \wedge \mathcal{P}^{\lozenge}_{\supset H}$$

This completes the proof.

□

3.3 Subgroups complexes and the Frattini subgroup

Definition 33. Let $F \subset G$ denote the intersection of all of the maximal proper subgroups of G. It is commonly referred to as the *Frattini subgroup.* When G is a p-group, F is the minimal subgroup of G such that G/F is an elementary abelian p-group.

Lemma 34. *If H does not contain the Frattini subgroup of G, then $\mathcal{P}(G)_{\supset H}$ is contractible.*

Proof. For any subgroup K of G, let KF denote the minimal subgroup of G containing both K and F. Consider the poset map

$$f : \mathcal{P}(G)_{\supset H} \to \mathcal{P}(G)_{\supset H}$$

$$K \mapsto KF$$

Any proper subgroup K is contained in some maximal subgroup M of G, and therefore the group KF is also contained in $M = MF$. Therefore, KF is also a proper subgroup of G and so f is a well defined map.

Since $KF \supset K$ for every K, the map f is homotopic to the identity. Because $KF \supset HF$ for every K, the map f is homotopic to the constant map at HF. It follows that $\mathcal{P}(G)_{\supset H}$ is contractible. □

Proposition 35. *Suppose G is a finite group and X is any pointed G-space. If H does not contain the Frattini subgroup of G, then $\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G} \wedge X) \simeq \star$. If H contains the Frattini subgroup of G, then*

$$\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G} \wedge X) \simeq \Sigma\mathcal{P}(G)^{\lozenge}_{\supset H} \wedge B(G/H)_+ \wedge X^G$$

Proof. By Proposition 32,

$$\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G} \wedge X) \simeq \Sigma(\mathcal{P}(G)^{\lozenge}_{\supset H})_{hW} \wedge X^G$$

If H does not contain the Frattini subgroup, then $\mathcal{P}(G)^{\lozenge}_{\supset H}$ has contractible underlying points and thus $(\mathcal{P}(G)^{\lozenge}_{\supset H})_{hW}$ is contractible. If H does contain the Frattini subgroup, then every subgroup containing H is normal, and thus W acts trivially on $\mathcal{P}(G)^{\lozenge}_{\supset H}$. Thus $(\mathcal{P}(G)^{\lozenge}_{\supset H})_{hW} \simeq \mathcal{P}(G)^{\lozenge}_{\supset H} \wedge BW_+$. $\square$

4 Products on Primitives

Let X and Y be pointed G-spaces. Consider the composition below, where the second map is inclusion of fixed points for the diagonal inclusion $G \subset G \times G$:

$$\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(X)^G \wedge \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(Y)^G \to \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(X \wedge Y)^{G\times G} \hookrightarrow \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(X \wedge Y)^G.$$

Specialize to the case $X = Y = S^{\infty\overline{\rho}_G}$. There is a homeomorphism of G-spaces

$$X \wedge Y = S^{\infty\overline{\rho}_G} \wedge S^{\infty\overline{\rho}_G} \cong S^{\infty\overline{\rho}_G}$$

given by interleaving the copies of the reduced regular representation $\overline{\rho}_G$. Thus, we obtain a product structure on the $\mathbb{Z}/p$-module $\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(S^{\infty\overline{\rho}_G})^G$: it is a $\mathbb{Z}/p$-algebra whose 0 element is the basepoint.

Let us assume that G is a p-group. The goal of this section is to analyze the product on $\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(S^{\infty\overline{\rho}_G})^G$ in terms of its free $\mathbb{Z}/p$-module basis which arises from Propositions 11 and 29:

$$\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(S^{\infty\overline{\rho}_G})^G \simeq \prod_{H\in\mathcal{C}} \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G})) \simeq \prod_{H\in\mathcal{C}} \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma\mathcal{P}(G)^{\lozenge}_{\supset H} \wedge B(G/H)_+). \tag{4.1}$$

Let $H, K \in \mathcal{C}$ (Definition 28). Then the H-factor smashed with the K-factor maps to the $(H \cap K)$-factor in the decomposition of Equation 4.1, in the following way. Let p^m denote the ratio $\frac{[G:H][G:K]}{[G:(H\cap K)]}$, where $m \geq 0$. The product on the basis elements $\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G})$ is the composition

$$\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G}) \wedge \mathrm{Pr}^{G/K}(S^{\infty\overline{\rho}_G}) \to \mathrm{Pr}^{\frac{G\times G}{H\times K}}(S^{\infty\overline{\rho}_G} \wedge S^{\infty\overline{\rho}_G}) \to \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\mathrm{Pr}^{G/(H\cap K)}(S^{\infty\overline{\rho}_G})). \tag{4.2}$$

The first map in Equation 4.2 is the 'product' for primitives (Definition 6) and the second is 'restriction' for primitives (Definition 12) along the diagonal inclusion $G \subset G \times G$.

Definition 36. Suppose that G is a p-group, and let $H, K \in \mathcal{C}$ (Definition 28). We say that H and K are *transverse* if the diagonal inclusion below is an isomorphism,

$$G/(H \cap K) \hookrightarrow G/H \times G/K.$$

Equivalently, H and K are transverse iff $\frac{[G:H][G:K]}{[G:(H\cap K)]} = 1$. If the diagonal inclusion is not an isomorphism, we say H and K are *nontransverse*.

In the case where H and K are transverse, the composition of Equation 4.2 can be described explicitly in terms of a natural product on subgroup complexes, and this is done in Proposition 39. If H and K are nontransverse the situation is more subtle, and for our purposes the following weaker result suffices. Consider the stable analogue of Equation 4.1, which results from Propositions 20 and 29:

$$\Phi^G(\mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma^{\infty G}S^0)) \simeq \bigvee_{H\in\mathcal{C}} \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \wedge \mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G}). \tag{4.3}$$

The stable version of Equation 4.2 (with the middle term omitted) is

$$\Sigma^{\infty}\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G}) \wedge \Sigma^{\infty}\mathrm{Pr}^{G/K}(S^{\infty\bar{\rho}_G}) \to \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \wedge \Sigma^{\infty}\mathrm{Pr}^{G/(H\cap K)}(S^{\infty\bar{\rho}_G}) \tag{4.4}$$

Definition 37. Let $\overline{\mathrm{Sp}}^n_{\mathbb{Z}/p}(-)$ denote the functor

$$\overline{\mathrm{Sp}}^n_{\mathbb{Z}/p}(-) := \mathrm{Sp}^n_{\mathbb{Z}/p}(-)/\mathrm{Sp}^{n-1}_{\mathbb{Z}/p}(-).$$

There is an obvious quotient map $\mathrm{Sp}^n_{\mathbb{Z}/p}(-) \to \overline{\mathrm{Sp}}^n_{\mathbb{Z}/p}(-)$.

Proposition 43 says that the following composition is zero on $\mathbb{F}_p$-homology:

$$\begin{tikzcd}
\Sigma^{\infty}\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G}) \wedge \Sigma^{\infty}\mathrm{Pr}^{G/K}(S^{\infty\bar{\rho}_G}) \arrow[r] \arrow[dr, dashed] & \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \wedge \Sigma^{\infty}\mathrm{Pr}^{G/(H\cap K)}(S^{\infty\bar{\rho}_G}) \arrow[d] \\
& \overline{\mathrm{Sp}}^{p^m}_{\mathbb{Z}/p}(\Sigma^{\infty}S^0) \wedge \Sigma^{\infty}\mathrm{Pr}^{G/(H\cap K)}(S^{\infty\bar{\rho}_G})
\end{tikzcd}$$

This requires some work to do, and is a result of the following more general lemma. Let V be an $\mathbb{F}_p$-vector space of rank m, viewed as an abelian group. Let X be a pointed V-space, and let $\ell \gg 0$ be a positive integer,. Then we prove (Lemma 42) that the composition below, where the first map is the restriction and the second is the quotient

$$\mathrm{Pr}^V(\Sigma^{\ell}X) \to \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\Sigma^{\ell}X) \to \overline{\mathrm{Sp}}^{p^m}_{\mathbb{Z}/p}(\Sigma^{\ell}X)$$

is zero on $\mathbb{F}_p$-homology in degrees $\ell, \ell+1, \ldots, p\ell-1$. We then take $X = \mathrm{Pr}^{G/(H\cap K)}(S^{\infty\bar{\rho}_G} \wedge S^{\infty\bar{\rho}_G})$ and $V = \frac{G/H\times G/K}{G/(H\cap K)}$, and let $\ell \to \infty$ to deduce Proposition 43.

4.1 Subgroup Complexes and Transverse Subgroups

Recall that $\mathcal{P}(G)_{\supset H}$ denotes the poset of proper subgroups of G which strictly contain H (Definition 27), and $\mathcal{P}(G)^{\lozenge}_{\supset H}$ denotes its unreduced suspension, regarded as a pointed space. We write $\hat{\mathcal{P}}(G)_{\supseteq H}$ to mean the poset of all subgroups of G which contain H (including both H and G itself). Then there is a map of posets

$$\hat{\mathcal{P}}(G)_{\supseteq H} \times \hat{\mathcal{P}}(G)_{\supseteq K} \to \hat{\mathcal{P}}(G)_{\supseteq H\cap K}$$

$$(H', K') \mapsto H' \cap K'$$

If H and K are transverse, then $H' \cap K' = H \cap K$ if and only if $H' = H$ and $K' = K$. Therefore, we obtain a product map on the unreduced join of the two spaces $\mathcal{P}(G)^{\lozenge}_{\supset H}$ and $\mathcal{P}(G)^{\lozenge}_{\supset K}$.

$$\mathcal{P}(G)^{\lozenge}_{\supset H} \star \mathcal{P}(G)^{\lozenge}_{\supset K} \to \mathcal{P}(G)^{\lozenge}_{\supset H \cap K}.$$

Note that the join of two spaces is the same as the suspension of their smash product.

Definition 38. Let G be a group, and let $H, K \trianglelefteq G$ be subgroups such that the diagonal inclusion $G/(H \cap K) \to G/H \times G/K$ is an isomorphism. The assignment $(H', K') \mapsto H' \cap K'$ yields a map of pointed spaces

$$\Sigma\mathcal{P}(G)^{\lozenge}_{\supset H} \wedge \Sigma\mathcal{P}(G)^{\lozenge}_{\supset K} \to \Sigma\mathcal{P}(G)^{\lozenge}_{\supset H \cap K},$$

which we call the *subgroup complex product.*

Proposition 39. *Let G be a p-group. Let $H, K \in \mathcal{C}$ be transverse subgroups of G. Under the equivalence*

$$\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G}) \simeq \Sigma\mathcal{P}(G)^{\lozenge}_{\supset H} \wedge B(G/H)_+$$

of Proposition 35, the composite map

$$\mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G}) \wedge \mathrm{Pr}^{G/K}(S^{\infty\overline{\rho}_G}) \to \mathrm{Pr}^{G/(H\cap K)}(S^{\infty\overline{\rho}_G})$$

in Equation 4.2 is give by the subgroup complex product, smashed with the equivalence

$$B(G/H)_+ \wedge B(G/K)_+ \to B(G/(H \cap K))_+$$

Proof. The fixed point space $S^{\infty\overline{\rho}_G^H}$ has a covering indexed over the poset $\mathcal{P}(G)_{\supseteq H}$. The cover element corresponding to the subgroup $H' \supseteq H$ is $S^{\infty\overline{\rho}_G^{H'}}$. The fixed point space $S^{\infty\overline{\rho}_G^K}$ has a similar covering. The behavior of the interleaving map

$$S^{\infty\overline{\rho}_G^H} \wedge S^{\infty\overline{\rho}_G^K} \subset S^{\infty\overline{\rho}_G^{H\cap K}} \wedge S^{\infty\overline{\rho}_G^{H\cap K}} \simeq S^{\infty\overline{\rho}_G^{H\cap K}}$$

with respect to this covering sends (H', K') to $H' \cap K'$. Because H and K are transverse, one has that $H' \cap K' = H \cap K$ if and only if $H' = H$ and $K' = K$. Therefore, the interleaving map descends to the associated graded

$$(S^{\infty\overline{\rho}_G^H} / \bigcup_{H' \supsetneq H} S^{\infty\overline{\rho}_G^{H'}}) \wedge (S^{\infty\overline{\rho}_G^K} / \bigcup_{K' \supsetneq K} S^{\infty\overline{\rho}_G^{K'}}) \to (S^{\infty\overline{\rho}_G^{H\cap K}} / \bigcup_{L \supsetneq H\cap K} S^{\infty\overline{\rho}_G^L})$$

and this map is given by the subgroup complex product

$$\Sigma\mathcal{P}(G)^{\lozenge}_{\supset H} \wedge \Sigma\mathcal{P}(G)^{\lozenge}_{\supset K} \to \Sigma\mathcal{P}(G)^{\lozenge}_{\supset H \cap K}.$$

The action of $G/(H \cap K)$ on each side is trivial, so upon applying $G/(H \cap K)$ homotopy orbits, we get the desired result.

$\square$

4.2 Cyclic Powers and Primitives

Let Y be a pointed space. The symmetric powers of Y are defined in terms of the symmetric group. We will define *mod p cyclic powers* in terms of elementary abelian p-groups. The mod p cyclic powers of Y will be used to describe the restriction map on primitives.

Definition 40. Let m be any positive integer, and let $V \cong (\mathbb{Z}/p)^m$. Let $\mathrm{Fun}(V, Y)$ be the space of maps from V (viewed as a discrete space) to Y, with an action of V given by

$$(vf)(w) = f(v^{-1}w) \qquad v, w \in V, f \in \mathrm{Fun}(V, Y).$$

For two functions $f, f' \in \mathrm{Fun}(V, Y)$, suppose that there are p points $v_1, \ldots, v_p \in V$ such that

$$f(v_1) = \cdots = f(v_p), \quad f'(v_1) = \cdots = f(v_p), \quad \text{and}$$

$$f(v) = f'(v) \qquad \forall v \in V - \{v_1, \ldots, v_p\}.$$

Then we write $f \sim f'$, and let $\sim$ denote the equivalence relation on $\mathrm{Fun}(V, Y)$ generated by the above conditions. The quotient $\frac{\mathrm{Fun}(V,Y)/\sim}{V}$ is the *V-th mod p cyclic power* of Y, and is denoted by $\mathrm{Cyc}^V_{\mathbb{Z}/p}(Y)$. There is an obvious quotient map $\mathrm{Cyc}^V_{\mathbb{Z}/p}(Y) \to \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(Y)$.

Note: The equivalence relation $\sim$ is what makes it 'mod p'. Without this relation, we obtain a construction $\mathrm{Fun}(V, Y)/V$ which we would call $\mathrm{Cyc}^V(Y)$ if we had any use for it.

Definition 41. Let $\mathrm{Fun}_0(V, Y) \subset \mathrm{Fun}(V, Y)$ denote the subspace consisting of those maps which send at least one vector v to the basepoint of Y, and let $\overline{\mathrm{Fun}}(V, Y) := \mathrm{Fun}(V, Y)/\mathrm{Fun}_0(V, Y)$. The quotient $\frac{\overline{\mathrm{Fun}}(V,Y)/\sim}{V}$ is called the *reduced V-th mod p cyclic power of* Y, and is denoted by $\overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(Y)$. There is a commutative square of

$$\begin{tikzcd}
\mathrm{Cyc}^V_{\mathbb{Z}/p}(Y) \arrow[r] \arrow[d] & \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(Y) \arrow[d] \\
\overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(Y) \arrow[r] & \overline{\mathrm{Sp}}^{p^m}_{\mathbb{Z}/p}(Y)
\end{tikzcd}.$$

Now suppose that Y is a pointed V-space, and let $Y^{\{e\}}$ denote its underlying points. Let $\varphi : Y \to \mathrm{Fun}(V, Y^{\{e\}})$ denote the V-equivariant map defined by $(\varphi(y))(v) = vy$. Let $0 \in \mathrm{Fun}(V, Y^{\{e\}})$ denote the function which sends every element of V to the basepoint of $Y^{\{e\}}$. It is easily seen that if $W \subset V$ is a nonzero subspace and $y \in Y^W$, then $\varphi(y) \sim 0$. Therefore, φ determines a map (of pointed spaces)

$$\overline{\varphi} : \mathrm{Pr}^V(Y) := \frac{Y / \bigcup\limits_{W \neq 0} Y^W}{V} \to \frac{\mathrm{Fun}(V, Y^{\{e\}}) / \sim}{V} =: \mathrm{Cyc}^V_{\mathbb{Z}/p}(Y^{\{e\}}) \tag{4.5}$$

Crucially for us, we have a commutative diagram where the composition along the top row is the restriction map of Definition 12:

$$\begin{tikzcd} \mathrm{Pr}^V(Y) \arrow[r, "\overline{\varphi}"] \arrow[dr, dashed] & \mathrm{Cyc}^V_{\mathbb{Z}/p}(Y^{\{e\}}) \arrow[r] \arrow[d] & \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(Y^{\{e\}}) \arrow[d] \\ & \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(Y^{\{e\}}) \arrow[r] & \overline{\mathrm{Sp}}^{p^m}_{\mathbb{Z}/p}(Y^{\{e\}}) \end{tikzcd} \tag{4.6}$$

The entire preceding discussion is natural in Y, i.e. all maps arise from natural transformations of functors from pointed V-spaces to pointed spaces. The following lemma describes the behavior of the dotted map in the stable range.

Lemma 42. *Let $V \cong (\mathbb{Z}/p)^m$, let X be a pointed space with an action of V, and let $\ell \gg 0$ be a positive integer. Then letting $Y = \Sigma^\ell X$ in Equation 4.6, the composition*

$$\mathrm{Pr}^V(\Sigma^\ell X) \to \mathrm{Cyc}^V_{\mathbb{Z}/p}(\Sigma^\ell X^{\{e\}}) \to \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(\Sigma^\ell X^{\{e\}})$$

is zero on reduced $\mathbb{F}_p$-homology in degrees $\ell, \ell+1, \dots, p\ell - 1$. Therefore, the map of spectra

$$\mathrm{Pr}^V(\Sigma^\infty X) \to \mathrm{Cyc}^V_{\mathbb{Z}/p}(\Sigma^\infty X^{\{e\}}) \to \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(\Sigma^\infty X)$$

is zero on $\mathbb{F}_p$-homology. And therefore, the composition of the restriction map (Definition 12) with the quotient map is zero on $\mathbb{F}_p$-homology:

$$\mathrm{Pr}^V(\Sigma^\infty X) \to \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\Sigma^\infty X^{\{e\}}) \to \overline{\mathrm{Sp}}^{p^m}_{\mathbb{Z}/p}(\Sigma^\infty X).$$

Proof. We first address the case where $\dim(V) = 1$, i.e. $V \cong \mathbb{Z}/p$. Let $v \in V$ denote a generator. Write $Y = \Sigma^\ell X$. Then the V-space $\mathrm{Fun}(V, Y^{\{e\}})$ can be identified with the Cartesian product $Y^{\times p}$ with the permutation action of the group $\mathbb{Z}/p$.

Let $\Delta : Y \to Y^{\times p}$ denote the diagonal map, and let $\Delta_{\mathrm{tw}} : Y \to Y^{\times p}$ denote the twisted diagonal map $\Delta_{\mathrm{tw}}(y) = (y, vy, v^2 y, \dots, v^{p-1} y)$. Consider the following commutative diagram of pointed spaces, where the maps $Y^V \to Y$ are the inclusion of the fixed points, and the columns are cofiber sequences.

$$\begin{tikzcd} Y^V \arrow[r] \arrow[d] & Y \arrow[r, equal] \arrow[d, "\Delta"] & Y \arrow[d, "\Delta"] \\ Y \arrow[r, "\Delta_{\mathrm{tw}}"] \arrow[d] & Y^{\times p} \arrow[r] \arrow[d] & Y^{\wedge p} \arrow[d] \\ Y/Y^V \arrow[r] & Y^{\times p}/\mathrm{im}(\Delta) \arrow[r] & Y^{\wedge p}/\mathrm{im}(\Delta) \end{tikzcd} \tag{4.7}$$

After quotienting by the free action of V, the bottom row of Diagram 4.7 is the composition

$$\mathrm{Pr}^V(Y) \to \mathrm{Cyc}^V_{\mathbb{Z}/p}(Y^{\{e\}}) \to \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(Y^{\{e\}}).$$

Therefore, it suffices to prove that the composition along the bottom row of Diagram 4.7 is zero on homology in the stable range.

The space $Y^{\wedge p}$ has homology concentrated in degree $p\ell$ and higher, and so it is stably contractible. Apply the functor $\tilde{H}_*(-)$ to Diagram 4.7 for any $\ell \leq * \leq p\ell - 1$, omitting the middle column and adding another row at the top to obtain

$$\begin{tikzcd}
\tilde{H}_{*+1}(Y/Y^V) \arrow[r, "h"] \arrow[d, "j"] & \tilde{H}_{*+1}(Y^{\wedge p}/\mathrm{im}(\Delta)) \arrow[d, equal] \\
\tilde{H}_*(Y^V) \arrow[r, "f"] \arrow[d, "g"] & \tilde{H}_*(Y) \arrow[d] \\
\tilde{H}_*(Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\tilde{H}_*(Y/Y^V) \arrow[r, "\Sigma h"] & \tilde{H}_*(Y^{\wedge p}/\mathrm{im}(\Delta)).
\end{tikzcd} \tag{4.8}$$

The maps f and g are equal, and $g \circ j = 0$. Thus, $f \circ j = 0$, and thus $h = 0$. Thus, $\Sigma h = 0$, as desired.

Now suppose $\dim(V) \geq 2$. Write $V \cong L \oplus W$ where $\dim(L) = 1$. There is an obvious natural transformation $\mathrm{Fun}(L, \mathrm{Fun}(W, -)) \to \mathrm{Fun}(V, -)$, and therefore a natural transformation

$$\mathrm{Cyc}^L_{\mathbb{Z}/p}(\mathrm{Cyc}^W_{\mathbb{Z}/p}(-)) \to \mathrm{Cyc}^V_{\mathbb{Z}/p}(-).$$

This natural transformation descends to the reduced cyclic powers $\overline{\mathrm{Cyc}}^L_{\mathbb{Z}/p}(\overline{\mathrm{Cyc}}^W_{\mathbb{Z}/p}(-)) \to \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(-)$. The composition

$$\mathrm{Pr}^V(-) \cong \mathrm{Pr}^L(\mathrm{Pr}^W(-)) \to \overline{\mathrm{Cyc}}^L_{\mathbb{Z}/p}(\mathrm{Pr}^W(-)) \to \overline{\mathrm{Cyc}}^L_{\mathbb{Z}/p}(\overline{\mathrm{Cyc}}^W_{\mathbb{Z}/p}(-)) \to \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(-)$$

is the restriction $\mathrm{Pr}^V(-) \to \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(-)$. By the dimension 1 case, the first step of this composition

$$\mathrm{Pr}^L(\mathrm{Pr}^W(\Sigma^\ell X)) = \mathrm{Pr}^L(\Sigma^\ell \mathrm{Pr}^W(X)) \to \overline{\mathrm{Cyc}}^L_{\mathbb{Z}/p}(\Sigma^\ell \mathrm{Pr}^W(X)) = \overline{\mathrm{Cyc}}^L_{\mathbb{Z}/p}(\mathrm{Pr}^W(\Sigma^\ell X))$$

is zero on $\mathbb{F}_p$-homology in degrees $\ell, \ell+1, \ldots, p\ell - 1$. Therefore, the composition $\mathrm{Pr}^V(\Sigma^\ell X) \to \overline{\mathrm{Cyc}}^V_{\mathbb{Z}/p}(\Sigma^\ell X)$ is zero on $\mathbb{F}_p$-homology in degrees $\ell, \ell+1, \ldots, p\ell - 1$.

$\square$

Proposition 43. *Let G be a p-group. Let $H, K \in \mathcal{C}$ be nontransverse subgroups of G. Let $p^m = \frac{[G:H][G:K]}{G:(H\cap K)]}$. Consider the following diagram where the horizontal map is the product of Diagram 4.4 and the vertical map is the quotient*

$$\begin{tikzcd}
\Sigma^\infty \mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G}) \wedge \Sigma^\infty \mathrm{Pr}^{G/K}(S^{\infty\overline{\rho}_G}) \arrow[r] \arrow[dr, dashed] & \mathrm{Sp}^{p^m}_{\mathbb{Z}/p}(\Sigma^\infty S^0) \wedge \Sigma^\infty \mathrm{Pr}^{G/(H\cap K)}(S^{\infty\overline{\rho}_G}) \arrow[d] \\
& \overline{\mathrm{Sp}}^{p^m}_{\mathbb{Z}/p}(\Sigma^\infty S^0) \wedge \Sigma^\infty \mathrm{Pr}^{G/(H\cap K)}(S^{\infty\overline{\rho}_G})
\end{tikzcd}.$$

The composition (dotted) is zero on $\mathbb{F}_p$-homology.

Proof. Take $X = \mathrm{Pr}^{G/(H\cap K)}(S^{\infty\overline{\rho}_G} \wedge S^{\infty\overline{\rho}_G})$, where G acts diagonally on $S^{\infty\overline{\rho}_G} \wedge S^{\infty\overline{\rho}_G}$. Let $V = \frac{G/H\times G/K}{G/(H\cap K)}$. Observe that X has a residual action of V, and

$$\mathrm{Pr}^V(\Sigma^\infty X) \simeq \Sigma^\infty \mathrm{Pr}^{G/H}(S^{\infty\overline{\rho}_G}) \wedge \Sigma^\infty \mathrm{Pr}^{G/K}(S^{\infty\overline{\rho}_G}).$$

The last line of Lemma 42 immediately implies the desired result. □

5 The first layer of the filtration

Let G be a p-group, and let X be any pointed G-space. Let $\mathrm{Aff}_1 \simeq (\mathbb{Z}/p) \rtimes \mathrm{GL}_1(\mathbb{F}_p)$ be the group consisting of affine transformation of the one-dimensional vector space over $\mathbb{F}_p$, and fix an inclusion $\mathrm{Aff}_1 \subseteq \Sigma_p$ into the symmetric group on p letters. The equivariant classifying space $B_G(-)$ is a functor from groups to G-space, and is constructed in Definition 45. The notion of a p-local equivalence of spectra is Definition 48. The main result of this section is the following proposition, which is proven here using several intermediate results to be proven in the body of this section.

Proposition 44. *There is a p-local equivalence of genuine G-spectra*

$$\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} X)/\mathrm{Sp}^1(\Sigma^{\infty G} X) \simeq X \wedge S^1 \wedge \Sigma^{\infty G}_+ B_G \mathrm{Aff}_1.$$

Note: The equivalence above is the $n = 1$ case of Theorem 1.

Proof. There is a cofiber sequence of functors

$$\mathrm{Sp}^1(-) \xrightarrow{\Delta} \mathrm{Sp}^p(-) \longrightarrow \mathrm{Sp}^p_{\mathbb{Z}/p}(-)$$

where Δ is the diagonal map. Thus by Lemma 16, there is a natural equivalence of genuine G-spectra

$$\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} X)/\mathrm{Sp}^1_{\mathbb{Z}/p}(\Sigma^{\infty G} X) \simeq X \wedge \mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^1_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0).$$

Proposition 49 implies that the quotient map is a p-local equivalence

$$\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G}S^0)/\mathrm{Sp}^1(\Sigma^{\infty G}S^0) \simeq \mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G}S^0)/\mathrm{Sp}^{p-1}(\Sigma^{\infty G}S^0).$$

Let $\mathcal{F}$ denote the family of nontransitive subgroups of Σ_p. Proposition 47 says that

$$\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G}S^0)/\mathrm{Sp}^{p-1}(\Sigma^{\infty G}S^0) \simeq \Sigma^{\infty G}(S^1 \wedge E_G\mathcal{F}_+/\Sigma_p).$$

Propositions 54 and 55 say that there are p-local equivalences of equivariant classifying spaces

$$E_G\mathcal{F}_+/\Sigma_p \simeq (B_G\Sigma_p)_+ \simeq (B_G\mathrm{Aff}_1)_+.$$

□

5.1 Equivariant Classifying Spaces

The following definition of a G-equivariant classifying space is given in [20].

Definition 45. Let Λ be any finite group. Suppose that $\mathcal{F}$ is a collection of subgroups of Λ with the property that if $\Gamma \in \mathcal{F}$, then every subgroup of Γ and every group conjugate to Γ is in $\mathcal{F}$. Then we define $E_G\mathcal{F}$ to be the $(G \times \Lambda)$-space with fixed points under any subgroup $\Gamma \subset (G \times \Lambda)$

$$(E_G\mathcal{F})^\Gamma \simeq \begin{cases} * & \text{if } \Gamma \cap \Lambda \in \mathcal{F} \\ \emptyset & \text{if } \Gamma \cap \Lambda \notin \mathcal{F} \end{cases}$$

When $\mathcal{F}$ contains only the trivial group, $E_G\mathcal{F}$ is denoted $E_G\Lambda$. This space has a free action of Λ. We call $B_G\Lambda = (E_G\Lambda)/\Lambda$ the *G-equivariant classifying space* of Λ.

Example 46. Let

- ρ_G be the real regular representation of G,
- $\overline{\mathbb{R}^n}$ be the reduced permutation representation of Σ_n,
- $\mathcal{F}$ denote the family of nontransitive subgroups of Σ_n, and
- for any representation V, let $U(V)$ denote the unit sphere of V.

Then Lemma 21 implies that $U(\infty\rho_G \otimes \overline{\mathbb{R}^n}) = E_G\mathcal{F}$.

Specialize the above example to the case $n = p$.

Proposition 47. *Let $\mathcal{F}$ denote the collection of nontransitive subgroups of Σ_p. The cofiber $\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G}S^0)/\mathrm{Sp}^{p-1}_{\mathbb{Z}/p}(\Sigma^{\infty G}S^0)$ is equivalent to to $\Sigma^{\infty G}(S^1 \wedge E_G\mathcal{F}_+/\Sigma_p)$.*

Proof. Let $X = S^V$, for any G-representation V. Then

$$\begin{aligned}
\mathrm{Sp}^p_{\mathbb{Z}/p}(S^V)/\mathrm{Sp}^{p-1}_{\mathbb{Z}/p}(S^V) &\simeq \mathrm{cof}(\, S^V \xrightarrow{\Delta} (S^V)^{\wedge p}/\Sigma_p\,) \\
&\simeq \mathrm{cof}(\, S^V \xrightarrow{\Delta} (S^V)^{\wedge p}\,)/\Sigma_p \\
&\simeq S^V \wedge \mathrm{cof}(\, S^0 \longrightarrow S^{V\otimes\overline{\mathbb{R}^p}}\,)/\Sigma_p \\
&\simeq S^V \wedge S^1 \wedge U(V \otimes \overline{\mathbb{R}^p})_+/\Sigma_p.
\end{aligned}$$

It therefore suffices for us to prove that $U(\infty(\rho_G \otimes \overline{\mathbb{R}^p})) \simeq E_G\mathcal{F}$. It is an easy consequence of Lemma 21 that for any subgroup $\Gamma \subset G \times \Sigma_p$,

$$U(\infty(\rho_G \otimes \overline{\mathbb{R}^p})) = \begin{cases} U(0) = \emptyset & \text{if} \quad \Gamma \cap \Sigma_p \quad \text{transitive} \\ U(\mathbb{R}^\infty) \simeq \star & \text{otherwise} \end{cases}$$

which completes the proof. □

5.2 p-local equivalences and Symmetric powers

It is well-known that the homotopy category of spectra carries a *p-localization* endofunctor $L_p(-)$, and for every spectra $\mathbf{X}$, there is a map $\mathbf{X} \to L_p\mathbf{X}$. For our purposes, here is the definition of the p-localization we need.

Definition 48. A spectrum $\mathbf{X}$ is *p-locally contractible* if $\tilde{H}_*(\mathbf{X};\mathbb{Z}_{(p)}) \cong 0$. A genuine G-spectrum $\mathbf{X}$ is p-locally contractible if $\Phi^H\mathbf{X}$ is p-locally contractible for every subgroup $H \subseteq G$. A map $f : \mathbf{X} \to \mathbf{Y}$ is a *p-local equivalence* if the cofiber is p-locally contractible.

Note that for CW complexes with finitely many cells in each dimension, a $\mathbb{Z}_{(p)}$-homology isomorphism is the same as an $\mathbb{F}_p$-homology isomorphism.

Proposition 49. *Let G be a p-group. The n-th layer $\mathrm{Sp}^n(\Sigma^{\infty G}S^0)/\mathrm{Sp}^{n-1}(\Sigma^{\infty G}S^0)$ in the symmetric powers of $\Sigma^{\infty G}S^0$ is p-locally contractible for $n = 2, 3, \ldots, p-1$.*

Proof. Pick some positive integer ℓ, and consider the n-th layer $\mathrm{Sp}^n(S^{\ell\rho_G})/\mathrm{Sp}^{n-1}(S^{\ell\rho_G}) \simeq (S^{\ell\rho_G})^{\wedge n}/\Sigma_n$. By induction on the group G, it is sufficient to prove that

$$\tilde{H}_*(((S^{\ell\rho_G})^{\wedge n}/\Sigma_n)^G; \mathbb{F}_p) = 0, \qquad * \le 2n.$$

Since $n < p$, there are no nontrivial homomorphisms $G \to \Sigma_n$, and therefore

$$((S^{\ell\rho_G})^{\wedge n}/\Sigma_n)^G = ((S^{\ell\rho_G})^{\wedge n})^G/\Sigma_n = (S^\ell)^{\wedge n}/\Sigma_n$$

As in the notation of Lemma 21, let $\overline{\mathbb{R}^n}$ denote the reduced standard representation of Σ_n. Then

$$(S^\ell)^{\wedge n}/\Sigma_n \simeq S^\ell \wedge S^{(\overline{\mathbb{R}^n})^\ell}/\Sigma_n$$

Since p is relatively prime to the order of Σ_n,

$$\tilde{H}_*(S^\ell \wedge S^{(\overline{\mathbb{R}^n})^\ell}/\Sigma_n; \mathbb{F}_p) \cong \tilde{H}_*(S^\ell \wedge S^{(\overline{\mathbb{R}^n})^\ell}; \mathbb{F}_p)_{\Sigma_n}$$

which is concentrated in degree ℓn and higher. Since $n \geq 2$, the result follows.

□

5.3 Nontransitive subgroups of Σ_p

Example 46 says that the $(G \times \Sigma_p)$-space $E_G\mathcal{F}$ is the space of ordered configurations of p (not necessarily distinct) points in $\infty\rho_G$ whose sum is zero and total length is 1. The subspace consisting of configurations of *distinct* points carries a free Σ_p action and is the $(G \times \Sigma_p)$-space $E_G\Sigma_p$.

In this section, we prove that if G is a p-group, then the inclusion $\Sigma^{\infty G}(B_G\Sigma_p)_+ \to \Sigma^{\infty G}(E_G\mathcal{F}_+/\Sigma_p)$ is a p-local equivalence (Proposition 54). The proof relies on Lemma 53, which establishes a formula for the G-fixed points of a quotient space.

We must first establish two simple lemmas which are used to prove Lemma 53.

Lemma 50. *Let $\sigma \in \Sigma_p$ be a p-cycle, and let $\Phi \subset \Sigma_p$ be a nontrivial group normalized by σ. Then Φ acts transitively on $\{1, \ldots, p\}$.*

Proof. Without loss of generality, let σ be the permutation sending $1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto p \mapsto 1$. Suppose that Ψ has some nontrivial permutation π sending $p \mapsto j$. Then $\sigma^{ij} j \pi \sigma^{-ij}$ sends $ij \mapsto (i+1)j$, for $i = 1, 2, \ldots, p-1$. Since Ψ is normalized by σ, these permutations $\sigma^{ij}\pi\sigma^{-ij}$ all lie in Ψ, and so Ψ is transitive. □

Lemma 51. *Let G be a p-group. If $f, f' : G \to \Sigma_p$ are two distinct maps, then the group $\Gamma \subset G \times \Sigma_p$ generated by Γ_f and $\Gamma_{f'}$ intersects Σ_p transitively.*

Proof. Pick some $g \in G$ such that $f(g) \neq f'(g)$, and moreover such that $f(g)$ is not the identity permutation. In particular, $f(g)$ must be a p-cycle because G is a p-group. Then Γ contains $(g^{-1}, f(g))(g, f'(g)) = f(g)^{-1}f'(g) \neq \text{id}$. Hence, Γ intersects Σ_p nontrivially. Let $\Psi = \Gamma \cap \Sigma_p$: then Φ is normalized by $f(g)$ because

$$(g, f(g))\Gamma(g^{-1}, f(g)^{-1}) = \Gamma \implies f(g)\Psi f(g)^{-1} = \Psi$$

Now using the fact that $f(g)$ is a p-cycle and Lemma 50, it follows that Φ is transitive. □

Definition 52. ([20], Definition 14) Let G and Λ be any finite groups, and let $H \subseteq G$ be a subgroup. For any homomorphism $f : H \to \Lambda$, its *graph* is the subgroup of $H \times \Lambda$

$$\Gamma_f := \{(h, f(h)) \ : \ h \in H\}.$$

Lemma 53. *Let G be a p-group, and let X be a $(G\times\Sigma_p)$-space. For each $x\in X$, let S_x denote the isotropy group of x. Suppose that for every point $x\in X$, the intersection $S_x\cap(1\times\Sigma_p)$ is nontransitive. Then there is a decomposition*

$$(X/\Sigma_p)^G = (\coprod_{f:G\to\Sigma_p} X^{\Gamma_f})/\Sigma_p \tag{5.1}$$

where f varies over all homomorphisms from G to Σ_p, and $\Gamma_f\subset(G\times\Sigma_p)$ denotes the graph of f. Here, a permutation σ in Σ_p takes a point of X^{Γ_f} to a point of $X^{\Gamma_{\sigma f\sigma^{-1}}}$.

Proof. Let x be a point in the $(G\times\Sigma_p)$-space X, and let us suppose that the image of x in X/Σ_p is G-fixed. This occurs if and only if the projection of S_x onto G is surjective.

We will show that S_x contains some graph subgroup. Let us denote $S_x\cap(1\times\Sigma_p)$ by Ψ. Then the group S_x may be thought of as the graph of a homomorphism $G\to N_G(\Psi)/\Psi$, where $N_G(\Psi)$ is the normalizer of Ψ in G.

By assumption, Ψ is nontransitive. If $\Psi=1$, then S_x automatically contains a graph subgroup. If Ψ is nontrivial, then by Lemma 50 the normalizer $N_G(\Psi)$ contains no p-cycles. Therefore, the order of the group $N_G(\Psi)/\Psi$ is not divisible by p. So there are no nontrivial homomorphisms from G to $N_G(\Psi)/\Psi$. It follows that $S_x=G\times\Psi$, and therefore S_x contains a graph subgroup. In either case, S_x contains some graph subgroup Γ_f, and thus $x\in X^{\Gamma_f}$. It follows that

$$(X/\Sigma_p)^G = (\bigcup_{f:G\to\Sigma_p} X^{\Gamma_f})/\Sigma_p.$$

All that remains is to show that if f and f' are two distinct homomorphisms from G to Σ_p, then X^{Γ_f} and $X^{\Gamma_{f'}}$ are disjoint. By the assumption on X, it suffices to show that the subgroup of $G\times\Sigma_p$ which is generated by Γ_f and $\Gamma_{f'}$ intersects $1\times\Sigma_p$ transitively. This is Lemma 51. ∎

Proposition 54. *The inclusion of G-spaces $(E_G\Sigma_p)/\Sigma_p\to(E_G\mathcal{F})/\Sigma_p$ is a p-local equivalence on all fixed point spaces.*

Proof. Let X denote the mapping cone of the inclusion $E_G\Sigma_p\to E_G\mathcal{F}$. By induction on the group G, it suffices to prove that $\tilde{H}_*((X/\Sigma_p)^G;\mathbb{F}_p)=0$. By Lemma 53,

$$(X/\Sigma_p)^G \simeq (\bigvee_{f:G\to\Sigma_p} X^{\Gamma_f})/\Sigma_p$$

But for an arbitrary subgroup $\Gamma\subset(G\times\Sigma_p)$,

$$X^\Gamma\simeq\begin{cases}\star & \text{if } \Gamma\cap\Sigma_p=1\\ S^0 & \text{if } \Gamma\cap\Sigma_p \text{ nontransitive and nonempty}\\ \star & \text{if } \Gamma\cap\Sigma_p \text{ transitive}\end{cases}$$

It therefore follows that $X^{\Gamma_f} \simeq \star$ for any graph subgroup Γ_f, and so $\tilde{C}_*(\bigvee_{f:G\to\Sigma_p} X^{\Gamma_f};\mathbb{F}_p)$ is acyclic. Any point in the pointed Σ_p-space $\bigvee_{f:G\to\Sigma_p} X^{\Gamma_f}$ has isotropy group nontransitive, and therefore $\tilde{C}_*(\bigvee_{f:G\to\Sigma_p} X^{\Gamma_f};\mathbb{F}_p)$ is a projective $\mathbb{F}_p[\Sigma_p]$-module. It then follows that $\tilde{C}_*((\bigvee_{f:G\to\Sigma_p} X^{\Gamma_f})/\Sigma_p;\mathbb{F}_p)$ is acyclic, as desired.

□

5.4 Equivalence of Classifying Spaces

We prove a proposition analogous to a well-known nonequivariant statement, namely that the map $B_G\mathrm{Aff}_1 \to B_G\Sigma_p$ is a p-local equivalence on all fixed point spaces.

Let G and Λ be groups, and let X be a $(G\times\Lambda)$-space. For any subgroup $\Psi \subseteq \Lambda$, let $C_\Lambda(\Psi)$ denote its centralizer. In ([20], Definition 14), the following formula is given

$$(X \times_\Lambda E_G\Lambda)^G \simeq \coprod_{[f]\in\mathrm{Hom}(G,\Lambda)/\Lambda} (X^{\Gamma_f})_{hC_\Lambda(\mathrm{im}f)}.$$

Note that this formula is a special case of Equation 5.1. Specializing to the case $X = \star$, we deduce that

$$(B_G\Lambda)^G \simeq \coprod_{[f]\in\mathrm{Hom}(G,\Lambda)/\Lambda} BC_\Lambda(\mathrm{im}f). \tag{5.2}$$

Proposition 55. *Let G be a p-group. The G-space map $B_G\mathrm{Aff}_1 \to B_G\Sigma_p$ induced by the inclusion $\iota : \mathrm{Aff}_1 \hookrightarrow \Sigma_p$ is a p-local equivalence.*

Proof. By induction on the group G, it suffices to check that the map of G-fixed points is an $\mathbb{F}_p$-homology isomorphism. The base case, $G = \{1\}$, is equivalent to proving that $B\iota : B\mathrm{Aff}_1 \to B\Sigma_p$ is a mod p homology equivalence. This is an immediate consequence of ([1], Theorem 5.5).

Let G be any p-group. By Equation 5.2, the map $(B_G\iota)^G : (B_G\mathrm{Aff}_1)^G \to (B_G\Sigma_p)^G$ is given by

$$(B_G\iota)^G : \coprod_{[f]\in\mathrm{Hom}(G,\mathrm{Aff}_1)/\mathrm{Aff}_1} BC_{\mathrm{Aff}_1}(\mathrm{im}f) \to \coprod_{[f]\in\mathrm{Hom}(G,\Sigma_p)/\Sigma_p} BC_{\Sigma_p}(\mathrm{im}f).$$

Every nontrivial homomorphism $G \to \mathrm{Aff}_1$ or $G \to \Sigma_p$ factors through a quotient of G isomorphic to $\mathbb{Z}/p$, so we may assume $G = \mathbb{Z}/p$. We now describe conjugacy classes of homomorphisms from $\mathbb{Z}/p$ to each of Aff_1 and Σ_p.

- The nontrivial homomorphisms $f : \mathbb{Z}/p \to \mathrm{Aff}_1$ are all conjugate, and each has centralizer equal to $\mathrm{im}(f)$. The trivial homomorphism has centralizer Aff_1.

- The nontrivial homomorphisms $f : \mathbb{Z}/p \to \Sigma_p$ are all conjugate, and each has centralizer equal to $\mathrm{im}(f)$. The trivial homomorphism has centralizer Σ_p.

Thus, the map $(B_{\mathbb{Z}/p}\iota)^{\mathbb{Z}/p} : (B_{\mathbb{Z}/p}\mathrm{Aff}_1)^{\mathbb{Z}/p} \to (B_{\mathbb{Z}/p}\Sigma_p)^{\mathbb{Z}/p}$ is given by

$$(B_{\mathbb{Z}/p}\iota)^{\mathbb{Z}/p} : B\mathbb{Z}/p \sqcup B\mathrm{Aff}_1 \to B\mathbb{Z}/p \sqcup B\Sigma_p,$$

The map on the first summand is the identity. The map on the second summand is $B\iota : B\mathrm{Aff}_1 \to B\Sigma_p$, which is an $\mathbb{F}_p$-homology isomorphism ([1], Theorem 5.5).

□

Note: Just as in the nonequivariant case, $\Sigma^{\infty G}(B_G\Sigma_p)_+$ is a stable summand of $\Sigma^{\infty G}(B_G\mathbb{Z}/p)_+$ with the inclusion map given by the transfer.

6 Mod p symmetric powers and Steinberg summands

Let G be a p-group.

Definition 56. For every $n \geq 0$, define the genuine G-spectrum $M_G(n)$ by

$$M_G(n) := S^{-n} \wedge \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0).$$

When G is the trivial group, we simply write $M(n)$. Note that for any $i, j \geq 0$, the product maps on mod p symmetric powers (Definition 4) give rise to product maps

$$M_G(i) \wedge M_G(j) \to M_G(i + j).$$

Definition 57. Recall from ([20], Definition 12) that for any pointed $(G \times \mathrm{GL}_n)$-space X, its *Steinberg summand* $e_n X$ is the naïve G-spectrum

$$e_n X := (\Sigma^{1-n}\mathbf{B}_n^\diamond \wedge X) \wedge_{\mathrm{GL}_n} (E_G\mathrm{GL}_n)_+.$$

Define the genuine G-spectrum $\mathbf{e}_n X$ by promoting $e_n X$ via Definition 15, i.e.

$$\mathbf{e}_n X := i_*(e_n X).$$

We construct an equivalence between the genuine G-spectrum $M_G(n)$ and the Steinberg summand of the equivariant classifying space $B_G(\mathbb{Z}/p)^n_+$, i.e.

Theorem 1. *For every integer $n \geq 1$, there is an equivalence of genuine G-spectra*

$$M_G(n) \simeq \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+.$$

We provide the proof here, referencing the supporting computational results proven in this section.

Proof. Proposition 44 states that there is an equivalence of genuine G-spectra

$$M_G(1) \simeq \Sigma_+^{\infty G} B_G \mathrm{Aff}_1.$$

Because $\mathbf{B}_1^\lozenge \simeq S^0$, it follows immediately from the definitions that there is an equivalence of genuine G-spectra $\mathbf{e}_1 B_G(\mathbb{Z}/p)_+ \simeq \Sigma^{\infty G}(B_G\mathrm{Aff}_1)_+$. Therefore, Proposition 44 is the $n = 1$ case of Theorem 1, namely

$$M_G(1) \simeq \mathbf{e}_1 B_G \mathbb{Z}/p_+. \tag{6.1}$$

The proof of Theorem 1 relies on the follow diagram

$$\mathbf{e}_n B_G(\mathbb{Z}/p)^n_+ \longrightarrow (\mathbf{e}_1 B_G \mathbb{Z}/p_+)^{\wedge n} = M_G(1)^{\wedge n} \longrightarrow M_G(n) \tag{6.2}$$

where the first map is the inclusion of the Steinberg summand, the middle equivalence is from Equation 6.1, and the last map is the product map. In diagram 6.2 above, we wish to show that the composition is a p-local equivalence on all geometric fixed point spectra. By induction on the group G, it will suffice to check that it is a p-local equivalence on G-geometric fixed points, and this can be checked at the level of $\mathbb{F}_p$-homology. That is, we wish to show that the induced map

$$f : H_*(\Phi^G(\mathbf{e}_n B_G(\mathbb{Z}/p)^n_+); \mathbb{F}_p) \to H_*(\Phi^G(M_G(n)); \mathbb{F}_p)$$

is an isomorphism of graded $\mathbb{F}_p$-vector spaces. This is Corollary 71. $\square$

The proof of Corollary 71 may be outlined as follows. The geometric fixed point spectra $\Phi^G(\mathbf{e}_n B_G(\mathbb{Z}/p)^n_+)$ and $\Phi^G(M_G(n))$ have been given explicit decompositions (discussed in Section 6.1, Proposition 60) indexed over the subgroups $H \in \mathcal{C}$. Thus in Section 6.3 we present bases for the two $\mathbb{F}_p$-vector spaces $H_*(\Phi^G(\mathbf{e}_n B_G(\mathbb{Z}/p)^n_+); \mathbb{F}_p)$ and $H_*(\Phi^G(M_G(n)); \mathbb{F}_p)$ and the map f all in terms of matrices. Because the two groups $H_*(\Phi^G(\mathbf{e}_n B_G(\mathbb{Z}/p)^n_+); \mathbb{F}_p)$ and $H_*(\Phi^G(M_G(n)); \mathbb{F}_p)$ are abstractly isomorphic, it suffices to prove that f has trivial kernel, which boils down to a linear algebra problem which is done in Section 6.2, Corollary 65.

6.1 H-summands

Let G be a p-group, and let $\mathcal{C}$ denote the poset of subgroups $H \trianglelefteq G$ such that G/H is an elementary abelian p-group (Definition 28).

Definition 58. For every $H \in \mathcal{C}$, let $d(H)$ denote the rank of G/H as an $\mathbb{F}_p$-vector space. Note that two subgroups $H, K \in \mathcal{C}$ are transverse (Definition 36) if and only if $d(H) + d(K) = d(H \cap K)$.

Proposition 20 implies that, for every $N \geq 1$ there is an equivalence of spectra

$$\Phi^G \mathrm{Sp}^N_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \simeq \bigvee_{H \in \mathcal{C}} \mathrm{Sp}^{\lfloor N/p^{d(H)} \rfloor}_{\mathbb{Z}/p}(\Sigma^\infty S^0) \wedge \mathrm{Pr}^{G/H}(S^{\infty \overline{\rho}_G})$$

which is suitably compatible with the inclusions $\mathrm{Sp}^{N-1}(-) \to \mathrm{Sp}^N(-)$. Taking $N = p^n$, we immediately deduce the formula

$$\begin{aligned}\Phi^G M_G(n) &\simeq \bigvee_{H \in \mathcal{C}} M_G(n - d(H)) \wedge \Sigma^{-d(H)} \mathrm{Pr}^{G/H}(S^{\infty \overline{\rho}_G}) \\ &\simeq \bigvee_{H \in \mathcal{C}} M_G(n - d(H)) \wedge \Sigma^{1-d(H)} \mathcal{P}(G)^{\Diamond}_{\supset H} \wedge B(G/H)_+\end{aligned} \tag{6.3}$$

Definition 59. Let n be a positive integer and $H \in \mathcal{C}$ be a subgroup of G. The spectrum

$$M_G(n, H) := M(n - d(H)) \wedge \Sigma^{-d(H)} \mathrm{Pr}^{G/H}(S^{\infty \overline{\rho}_G})$$

is called the *H-summand of $\Phi^G M_G(n)$*. For any two positive integers m, n and subgroups $H, K \in \mathcal{C}$, there is a product map

$$M_G(m, H) \wedge M_G(n, K) \to M_G(m + n, H \cap K)$$

which is determined by the product maps

$$\Phi^G M_G(m) \wedge \Phi^G M_G(n) \to \Phi^G M_G(m + n).$$

Proposition 60. *Let G be a p-group and let n be any positive integer. For every subgroup $H \in \mathcal{C}$, there is an equivalence of H-summand spectra*

$$M_G(n, H) \simeq E_n(H). \tag{6.4}$$

where $E_n(H)$ is the H-summand of the fixed point spectrum $(e_n B_G(\mathbb{Z}/p)^n_+)^G$ ([20], Definition 20).

Let m, n be positive integers and suppose that $H, K \in \mathcal{C}$ are transverse. Then there is a commutative diagram

$$\begin{tikzcd} M_{G,m}(H) \wedge M_{G,n}(K) \arrow[r] \arrow[d, equal] & M_{G,m+n}(H \cap K) \arrow[d, equal] \\ E_m(H) \wedge E_n(K) \arrow[r] & E_{m+n}(H \cap K) \end{tikzcd}$$

where the rows are the products on summands, and the columns are the equivalences just described.

Proof. By Proposition 35,

$$\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G}) \simeq \Sigma\mathcal{P}(G)^{\Diamond}_{\supset H} \wedge B(G/H)_+.$$

When G/H is elementary abelian, the subgroup complex $\mathcal{P}(G)_{\supset H}$ is identical to the flag complex of the $\mathbb{F}_p$-vector space $G/H \cong (\mathbb{Z}/p)^{d(H)}$. By ([18], Theorem A), the layer $M(n-d(H))$ is p-locally equivalent to $e_{n-d(H)}B(\mathbb{Z}/p)_+^{n-d(H)}$. Thus,

$$\begin{aligned} M_G(n,H) &:= M(n-d(H)) \wedge \Sigma^{-d(H)}\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G}) \\ &\simeq e_{n-d(H)}B(\mathbb{Z}/p)_+^{n-d(H)} \wedge \Sigma^{1-d(H)}\mathbf{B}^{\Diamond}_{d(H)} \wedge B(G/H)_+ \\ &=: E_n(H). \end{aligned}$$

When H and K are transverse, the subgroup complex product (Definition 38) is identical to the flag complex product ([20], above Proposition 10). Therefore, by combining Proposition 39 with ([20], Proposition 21), we deduce that the equivalence $\Sigma^{-d(H)}\mathrm{Pr}^{G/H}(S^{\infty\bar{\rho}_G}) \simeq \Sigma^{1-d(H)}\mathbf{B}^{\Diamond}_{n-d(H)} \wedge B(G/H)_+$ respects the products on both sides.

The equivalence $M(n-d(H)) \simeq e_{n-d(H)}B(\mathbb{Z}/p)_+^{n-d(H)}$ constructed in ([18], Theorem A) was built so as to respect the product structures on both sides. Therefore, the equivalence $M_G(n,H) \simeq E_n(H)$ respects the product structures on both sides. □

We note the following corollary.

Corollary 61. *There is an equivalence of spectra*

$$\Phi^G M_G(n) \simeq (e_n B_G(\mathbb{Z}/p)^n_+)^G.$$

Proof. Combine Equation 6.4, Equation 6.3, and ([20], Definition 20). □

6.2 Matrices with transverse row nullspaces

Definition 62. Let $n, r \geq 0$ be nonnegative integers. Then we write $\mathrm{Mat}_{n,r} = \mathrm{Hom}((\mathbb{Z}/p)^r, (\mathbb{Z}/p)^n)$ for the set of $n \times r$ matrices with entries in the field $\mathbb{F}_p$. For each subspace $V \subseteq (\mathbb{Z}/p)^r$, let $\mathrm{Mat}_{n,r}(V) \subset \mathrm{Mat}_{n,r}$ denote the set of $n \times r$ matrices with nullspace V.

Definition 63. Let $\mathcal{T} \subset \mathrm{Mat}_{n,r}$ denote the set of $n \times r$ matrices with the following property: if a matrix $A \in \mathcal{T}$ has exactly k nonzero rows for some k, then those nonzero row vectors are linearly independent. Let $\mathcal{T}(V) \subset \mathrm{Mat}_{n,r}(V)$ denote the intersection $\mathcal{T}(V) = \mathcal{T} \cap \mathrm{Mat}_{n,r}(V)$. The set $\mathcal{T}(V)$ is equivalently characterized as the set of $n \times r$ matrices with nullspace V and exactly s nonzero rows, where $\dim(V) = r - s$.

The GL_n-set $\mathrm{Mat}_{n,r}$ decomposes as

$$\mathrm{Mat}_{n,r} = \coprod_{V \subset (\mathbb{Z}/p)^r} \mathrm{Mat}_{n,r}(V).$$

Fix a subspace $V \subseteq (\mathbb{Z}/p)^r$, and write $\dim(V) = r - s$. Our goal in this section is to prove the Proposition 64 below.

Proposition 64. *Let D be any finite-dimensional $\mathbb{F}_p[\mathrm{GL}_n]$-module. Let A denote the $\mathbb{F}_p[\mathrm{GL}_n]$-module $A = \bigoplus_{\mathrm{Mat}_{n,r}(V)} D$. The composition*

$$\begin{tikzcd} e_n A \arrow[r, hook] & A \arrow[r, "\mathrm{proj}_{\mathcal{T}(V)}"] & A \arrow[r, "e_n(-)"] & e_n A \end{tikzcd}$$

is a monomorphism of $\mathbb{F}_p$-vector spaces. Therefore, because the source and target have the same dimension, the composition above is an isomorphism.

Before we discuss the proof of Proposition 64, we record a corollary, which is used in the proof of Corollary 71 and thus of Theorem 1.

Corollary 65. *Let D denote the $\mathbb{F}_p[\mathrm{GL}_n]$ module $D = H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p)$. The following composition is an isomorphism*

$$\begin{tikzcd} e_n \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[r, hook] & e_1^{\boxtimes n} \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[r, "\mathrm{proj}_{\mathcal{T}}"] & e_1^{\boxtimes n} \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[r] & e_n \bigoplus_{\mathrm{Mat}_{n,r}} D \end{tikzcd}.$$

Proof. The decomposition $\mathrm{Mat}_{n,r} = \bigsqcup_{V \subset (\mathbb{Z}/p)^r} \mathrm{Mat}_{n,r}(V)$ is GL_n-equivariant, so it suffices to prove the same statement with $\mathrm{Mat}_{n,r}$ replaced by $\mathrm{Mat}_{n,r}(V)$. Next, we observe that the following diagram commutes:

$$\begin{tikzcd} e_n \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[r, hook] \arrow[dr, hook] & \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[r, "\mathrm{proj}_{\mathcal{T}}"] & \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[r] & e_n \bigoplus_{\mathrm{Mat}_{n,r}} D \\ & e_1^{\boxtimes n} \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[u, hook] \arrow[r, "\mathrm{proj}_{\mathcal{T}}"] & e_1^{\boxtimes n} \bigoplus_{\mathrm{Mat}_{n,r}} D \arrow[u, hook] \arrow[ur] & \end{tikzcd}.$$

Now the proof statement is an immediate corollary of ([20], Proposition 64). $\square$

We now prove Proposition 64. We need some notation and lemmas.

Definition 66. Let $\Upsilon \subset \mathcal{T}(V)$ denote the subset of matrices whose last s rows are nonzero. Let N be the free $\mathbb{F}_p[\mathrm{GL}_n]$-module

$$N = \bigoplus_{\mathrm{Mat}_{n,r}(V)} \mathbb{F}_p[\mathrm{GL}_n].$$

Associated to the subsets $\Upsilon \subset \mathcal{T}(V) \subset \mathrm{Mat}_{n,r}(V)$ are endomorphisms $\mathrm{proj}_{\mathcal{T}(V)}$ and proj_{Υ} of the vector space N such that

$$\mathrm{proj}_{\mathcal{T}(V)} \circ \mathrm{proj}_{\Upsilon} = \mathrm{proj}_{\Upsilon} = \mathrm{proj}_{\Upsilon} \circ \mathrm{proj}_{\mathcal{T}(V)}.$$

Recall from ([20], Section 1) that B_n (resp. Σ_n) denotes the group of invertible $n\times n$ upper triangular matrices (resp. permutation matrices). The elements $\overline{B}_n$ and $\overline{\Sigma}_n$ of the group algebra $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$ define endomorphisms of the $\mathbb{F}_p[\mathrm{GL}_n]$-module N. The conjugate Steinberg idempotent $\hat{e}_n$ is the element of the group ring $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$ defined by $\hat{e}_n = \frac{1}{c_n}\cdot\overline{B}_n\overline{\Sigma}_n$, where $c_n\in\mathbb{Z}_{(p)}^{\times}$.

Let $B_{n-s}\times B_s\subset B_n$ (resp. $\Sigma_{n-s}\times\Sigma_s$) denote the set of upper-triangular matrices (resp. permutation matrices) whose ij-th entry is zero whenever $1\le i\le n-s$ and $n-s+1\le j\le n$. Let $\hat{e}_{n-s}\boxtimes\hat{e}_s\in\mathbb{F}_p[\mathrm{GL}_n]$ denote the image of the idempotent element $\hat{e}_{n-s}\otimes\hat{e}_s\in\mathbb{F}_p[\mathrm{GL}_{n-s}]\otimes_{\mathbb{F}_p}\mathbb{F}_p[\mathrm{GL}_s]$ under the block inclusion.

Lemma 67. *The following two endomorphisms of N as an $\mathbb{F}_p$-vector space are equal:*

$$\mathrm{proj}_\Upsilon\circ\overline{B}_n\circ\mathrm{proj}_{\mathcal{T}(V)} = \overline{B_{n-s}\times B_s}\circ\mathrm{proj}_\Upsilon.$$

Proof. Let A be an $n\times r$ matrix in $\mathcal{T}(V)$ and let b be an $n\times n$ invertible upper triangular matrix. Then

$$bA\in\Upsilon \iff (b\in B_{n-s}\times B_s \text{ and } A\in\Upsilon).$$

It immediately follows that

$$\mathrm{proj}_\Upsilon\circ\overline{B}_n\circ\mathrm{proj}_{\mathcal{T}(V)} = \mathrm{proj}_\Upsilon\circ\overline{B_{n-s}\times B_s}\circ\mathrm{proj}_{\mathcal{T}(V)}.$$

The set Υ is preserved by the group $\mathrm{GL}_{n-s}\times\mathrm{GL}_s$, and so it follows that

$$\begin{aligned}\mathrm{proj}_\Upsilon\circ\overline{B_{n-s}\times B_s}\circ\mathrm{proj}_{\mathcal{T}(V)} &= \overline{B_{n-s}\times B_s}\circ\mathrm{proj}_\Upsilon\circ\mathrm{proj}_{\mathcal{T}(V)}\\ &= \overline{B_{n-s}\times B_s}\circ\mathrm{proj}_\Upsilon.\end{aligned}$$

$\square$

Lemma 68. *The composition $\hat{e}_nN\hookrightarrow N\xrightarrow{\mathrm{proj}_\Upsilon}N$ is a monomorphism of $\mathbb{F}_p$-vector spaces.*

Proof. For any sub-$\mathbb{F}_p$-vector space $L\subseteq N$, we write $\mathrm{proj}_\Upsilon(L)$ to denote the image of L under the endomorphism proj_Υ. There is an inclusion of $\mathbb{F}_p$-vector spaces

$$\mathrm{proj}_\Upsilon(\hat{e}_nN)\subseteq\mathrm{proj}_\Upsilon((\hat{e}_{n-s}\boxtimes\hat{e}_s)N).$$

We will prove that these two vector spaces are equal. Because the set $\Gamma\subset\mathrm{Mat}_{n,r}(V)$ is preserved by the subgroup $\mathrm{GL}_{n-s}\times\mathrm{GL}_s\subset\mathrm{GL}_n$, it follows that

$$\mathrm{proj}_\Upsilon((\hat{e}_{n-s}\boxtimes\hat{e}_s)N) = \mathrm{proj}_\Upsilon\mathrm{proj}_\Upsilon((\hat{e}_{n-s}\boxtimes\hat{e}_s)N) = \mathrm{proj}_\Upsilon((\hat{e}_{n-s}\boxtimes\hat{e}_s)\mathrm{proj}_\Upsilon N).$$

Therefore the $\mathbb{F}_p$-vector space $\mathrm{proj}_\Upsilon((\hat{e}_{n-s}\boxtimes\hat{e}_s)N)$ is spanned by elements of the form

$$\mathrm{proj}_\Upsilon((\hat{e}_{n-s}\boxtimes\hat{e}_s)(A\otimes x)),\qquad A\in\Upsilon, x\in\mathrm{GL}_n.$$

Pick such an element. Let b be any element of the group of block upper triangular matrices $B_{n-s} \times B_s$. Then for every $\sigma \in \Sigma_n$, we have

$$\sigma \cdot b \cdot A \in \Upsilon \iff \sigma \in \Sigma_{n-s} \times \Sigma_s.$$

Therefore, it follows that

$$\mathrm{proj}_\Upsilon(\overline{\Sigma_{n-s} \times \Sigma_s} \cdot \overline{B_{n-s} \times B_s} \cdot (A \otimes x)) = \mathrm{proj}_\Upsilon(\overline{\Sigma}_n \cdot \overline{B_{n-s} \times B_s} \cdot (A \otimes x)).$$

Now let b be any element of the group B_n such that $b \notin B_{n-s} \times B_s$. Then it follows that $b \cdot A \notin \mathcal{T}(V)$. So for any permutation matrix $\sigma \in \Sigma_n$, we have $\sigma \cdot b \cdot A \notin \mathcal{T}(V)$. In particular, $\sigma \cdot b \cdot A \notin \Upsilon$. Therefore

$$\mathrm{proj}_\Upsilon(\overline{\Sigma}_n \cdot \overline{B_{n-s} \times B_s} \cdot (A \otimes x)) = \mathrm{proj}_\Upsilon(\overline{\Sigma}_n \cdot \overline{B}_n \cdot (A \otimes x)) = \mathrm{proj}_\Upsilon(\hat{e}_n(A \otimes x)).$$

We therefore conclude that

$$\mathrm{proj}_\Upsilon((\hat{e}_{n-s} \boxtimes \hat{e}_s)N) \subseteq \mathrm{proj}_\Upsilon(\hat{e}_n N),$$

and therefore the two $\mathbb{F}_p$-vector spaces above are equal.

We now prove that $\dim_{\mathbb{F}_p}(\hat{e}_n N) = \dim_{\mathbb{F}_p}(\mathrm{proj}_\Upsilon((\hat{e}_{n-s} \boxtimes \hat{e}_s)N))$. From this fact, we will conclude that $\hat{e}_n N$ and $\mathrm{proj}_\Upsilon(\hat{e}_n N)$ have the same dimension as $\mathbb{F}_p$-vector spaces, which completes the proof. Because $N = \bigoplus_{\mathrm{Mat}_{n,r}(V)} \mathbb{F}_p[\mathrm{GL}_n]$ is a free $\mathbb{F}_p[\mathrm{GL}_n]$-module, we calculate

$$\begin{aligned}
\dim_{\mathbb{F}_p}(\hat{e}_n N) = \dim_{\mathbb{F}_p}(e_n N) &= \dim_{\mathbb{F}_p}(\mathrm{St}_n \otimes_{\mathrm{GL}_n} N) \\
&= \dim_{\mathbb{F}_p}(\mathrm{St}_n) \cdot |\mathrm{Mat}_{n,r}(V)| \\
&= \boxed{p^{\binom{n}{2}} \prod_{i=0}^{s-1} (p^n - p^i)}.
\end{aligned}$$

Because $\mathrm{proj}_\Upsilon N = \bigoplus_\Upsilon \mathbb{F}_p[\mathrm{GL}_n]$ is a free $\mathbb{F}_p[\mathrm{GL}_{n-s} \times \mathrm{GL}_s]$-module, we calculate

$$\begin{aligned}
\dim_{\mathbb{F}_p}(\mathrm{proj}_\Upsilon((\hat{e}_{n-s} \boxtimes \hat{e}_s)N)) &= \dim_{\mathbb{F}_p}((\hat{e}_{n-s} \boxtimes \hat{e}_s)\mathrm{proj}_\Upsilon N) \\
&= \dim_{\mathbb{F}_p}(\mathrm{St}_{n-s} \otimes \mathrm{St}_s) \cdot |\Upsilon| \cdot \frac{|\mathrm{GL}_n|}{|\mathrm{GL}_{n-s} \times \mathrm{GL}_s|} \\
&= \dim_{\mathbb{F}_p}(\mathrm{St}_{n-s} \otimes \mathrm{St}_s) \cdot |\mathrm{GL}_s| \cdot \frac{|\mathrm{GL}_n|}{|\mathrm{GL}_{n-s} \times \mathrm{GL}_s|} \\
&= \boxed{p^{\binom{n-s}{2} + \binom{s}{2}} \cdot \frac{\prod_{i=0}^{n-1} (p^n - p^i)}{\prod_{i=0}^{n-s-1} (p^{n-s} - p^i)}}.
\end{aligned}$$

It is routine to check that the two boxed expressions are equal. ☐

Proof of Proposition 64. Any such D receives a surjective map from a finite-dimensional free $\mathbb{F}_p[\mathrm{GL}_n]$-module. Therefore it suffices to consider the case $D = \mathbb{F}_p[\mathrm{GL}_n]$. In this case, A is equal to the module $N = \bigoplus_{\mathrm{Mat}_{n,r}(V)} \mathbb{F}_p[\mathrm{GL}_n]$ defined earlier.

Let $x \in N$ be any element such that $e_n x$ is nonzero. We wish to prove that $e_n(\mathrm{proj}_{\mathcal{T}(V)}(e_n x))$ is nonzero. We will prove that $\mathrm{proj}_\Upsilon(e_n(\mathrm{proj}_{\mathcal{T}(V)}(e_n x)))$ is nonzero.

$$\begin{aligned}
\mathrm{proj}_\Upsilon(e_n(\mathrm{proj}_{\mathcal{T}(V)}(e_n x))) &:= \mathrm{proj}_\Upsilon \overline{B}_n \overline{\Sigma}_n \mathrm{proj}_{\mathcal{T}(V)} \overline{B}_n \overline{\Sigma}_n x \\
&= \mathrm{proj}_\Upsilon \overline{B}_n \mathrm{proj}_{\mathcal{T}(V)} \overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x \qquad (\mathcal{T}(V) \text{ is } \Sigma_n\text{–invariant}) \\
&= \overline{B_{n-s} \times B_s} \mathrm{proj}_\Upsilon \overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x. \qquad (\text{Lemma 3.3})
\end{aligned}$$

Recall from **??** that for any $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-module M, the linear operators

$$\overline{B}_n : \ \hat{e}_n M \rightleftarrows e_n M \ : \overline{\Sigma}_n$$

are inverse isomorphisms. Therefore, because the element $e_n x = \overline{B}_n \overline{\Sigma}_n x$ was assumed to be nonzero, it follows that the element $\overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x$ is nonzero. Therefore, by Lemma 3.2, the element $\mathrm{proj}_\Upsilon \overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x$ is nonzero.

The conjugate Steinberg idempotent $\hat{e}_n$ can be written in the form

$$\hat{e}_n = (\hat{e}_{n-s} \boxtimes \hat{e}_s) \overline{U}_{n-s,s} \overline{\Sigma_{\mathrm{shuf}}(n-s,s)}.$$

It therefore follows that the summand $\hat{e}_n N = \overline{\Sigma}_n \overline{B}_n N$ is a sub-$\mathbb{F}_p$-vector space of $(\hat{e}_{n-s} \boxtimes \hat{e}_s)N$. Therefore, the element $\overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x$ is an element of the vector space $(\hat{e}_{n-s} \boxtimes \hat{e}_s)N$. The set Υ is preserved by the group $\mathrm{GL}_{n-s} \times \mathrm{GL}_s$, and so it follows that the endomorphisms proj_Υ and $\hat{e}_{n-s} \boxtimes \hat{e}_s$ commute. Therefore, $\mathrm{proj}_\Upsilon \overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x$ lies in the vector space $(\hat{e}_{n-s} \boxtimes \hat{e}_s)N$.

The linear operator

$$\overline{B_{n-s} \times B_s} : (\hat{e}_{n-s} \boxtimes \hat{e}_s)N \to (e_{n-s} \boxtimes e_s)N$$

is an isomorphism. Therefore, because $\mathrm{proj}_\Upsilon \overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x$ is a nonzero element of the vector space $(\hat{e}_{n-s} \boxtimes \hat{e}_s)N$, we conclude that $\overline{B_{n-s} \times B_s} \mathrm{proj}_\Upsilon \overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n x$ is nonzero, as desired. □

6.3 Proof of Theorem 1

Let G be a p-group, and let n, i be positive integers such that $n \geq i$. For any pointed $(G \times \mathrm{GL}_n)$-space X, there are inclusions and projections of Steinberg summands as defined in ([20], Sections 2.2 and 2.3),

$$e_n X \to (e_i \boxtimes e_{n-i})X \quad \text{and} \quad (e_i \boxtimes e_{n-i})X \to e_n X.$$

We thus obtain an inclusion of naïve G-spectra $e_n X \to e_1^{\boxtimes n} X$.

We now specialize to the case where X is the G-equivariant classifying space with a disjoint basepoint, $X = B_G(\mathbb{Z}/p)^n_+$. In this situation, the fixed point spectrum $(e_n B_G(\mathbb{Z}/p)^n_+)^G$ was given a complete description in ([20], Section 4.2). We express that description in terms of matrices for the purposes of our computation.

Definition 69. Let G be a finite p-group. We denote by F the minimal subgroup of G such that G/F is elementary abelian. We let r denote the rank of G/F, and we fix an isomorphism $G/F \cong (\mathbb{Z}/p)^r$. The subgroups $H \in \mathcal{C}$ are in bijective correspondence with linear subspaces $V \subseteq (\mathbb{Z}/p)^r$, and we let $\mathrm{cd}(V)$ denote the codimension of V.

Thus,

$$\begin{aligned}(e_n B_G(\mathbb{Z}/p)^n_+)^G &\simeq e_n \bigvee_{\mathrm{Hom}(G,(\mathbb{Z}/p)^n)} B(\mathbb{Z}/p)^n_+ \\ &\simeq e_n \bigvee_{\mathrm{Mat}_{n,r}} B(\mathbb{Z}/p)^n_+\end{aligned} \tag{6.5}$$

There is an equivalence of $(\mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1)$-sets given by using the i-th input row vector as the i-th row of the output matrix for $i = 1, 2, \ldots, n$,

$$\mathrm{Mat}_{1,r} \times \cdots \times \mathrm{Mat}_{1,r} \cong \mathrm{Mat}_{n,r}.$$

Then the Steinberg inclusion $(e_n B_G(\mathbb{Z}/p)^n_+)^G \to (e_1^{\boxtimes n} B_G(\mathbb{Z}/p)^n_+)^G$ is described by the inclusion of Steinberg summands

$$e_n \bigvee_{\mathrm{Mat}_{n,r}} B(\mathbb{Z}/p)^n_+ \to e_1^{\boxtimes} \bigvee_{\mathrm{Mat}_{n,r}} B(\mathbb{Z}/p)^n_+ \simeq (e_1 \bigvee_{\mathrm{Mat}_{1,r}} B(\mathbb{Z}/p)_+)^{\wedge n}.$$

Expressed on the level of homology, the map $\tilde{H}_*(\Phi^G \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+; \mathbb{F}_p) \to \tilde{H}_*(\Phi^G \mathbf{e}_1^{\boxtimes} B_G(\mathbb{Z}/p)^n_+; \mathbb{F}_p)$ is given by the inclusion

$$e_n \bigoplus_{\mathrm{Mat}_{n,r}} H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \to e_1^{\boxtimes} \bigoplus_{\mathrm{Mat}_{n,r}} H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p). \tag{6.6}$$

The product $\tilde{H}_*(\Phi^G M_G(1)^{\wedge n}) \to \tilde{H}_*(\Phi^G M_G(n))$ can also be described in this language. By Proposition 44,

$$\begin{aligned}\tilde{H}_*(\Phi^G M_G(1)^{\wedge n}) &\cong \tilde{H}_*(\Phi^G \mathbf{e}_1^{\boxtimes n} B_G(\mathbb{Z}/p)^n_+; \mathbb{F}_p) \\ &\cong e_1^{\boxtimes} \bigoplus_{\mathrm{Mat}_{n,r}} H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p)\end{aligned} \tag{6.7}$$

And by Corollary 61,

$$\begin{aligned}\tilde{H}_*(\Phi^G M_G(n)) &\cong \tilde{H}_*(\Phi^G \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+; \mathbb{F}_p) \\ &\cong e_n \bigoplus_{\mathrm{Mat}_{n,r}} H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p)\end{aligned} \tag{6.8}$$

Proposition 70. *Under the isomorphisms of Equations 6.7 and 6.8, the product* $\tilde{H}_*(\Phi^G M_G(1)^{\wedge n}) \to \tilde{H}_*(\Phi^G M_G(n))$ *on the layers in the mod p symmetric powers is given by the composition*

$$e_1^{\boxtimes n} \bigoplus_{\mathrm{Mat}_{n,r}} H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \xrightarrow{\mathrm{proj}_{\mathcal{T}}} e_1^{\boxtimes n} \bigoplus_{\mathrm{Mat}_{n,r}} H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \longrightarrow e_n \bigoplus_{\mathrm{Mat}_{n,r}} H_*(B(\mathbb{Z}/p)^n; \mathbb{F}_p)\ .$$

Proof. The GL_1-set $\mathrm{Mat}_{1,r}$ is a disjoint union $\mathrm{Mat}_{1,r} = \coprod_{V\subseteq(\mathbb{Z}/p)^r} \mathrm{Mat}_{1,r}(V)$, where $\mathrm{Mat}_{1,r}(V)$ is the set of r-dimensional row vector with nullspace V. Note that $\mathrm{Mat}_{1,r}(V)$ is empty unless V has codimension 0 or 1. Let $H_1,\dots,H_n \in \mathcal{C}$, and let $V_1,\dots,V_n \subseteq (\mathbb{Z}/p)^r$ be the associated subspaces. There is an isomorphism by using the i-th input row vector as the i-th row of the output matrix for $i = 1, 2, \dots, n$,

$$\coprod_{V_1,\dots,V_n\subseteq(\mathbb{Z}/p)^r} \mathrm{Mat}_{1,r}(V_1)\times\cdots\times\mathrm{Mat}_{1,r}(V_n) \cong \coprod_{V\subseteq(\mathbb{Z}/p)^r} \mathrm{Mat}_{n,r}(V).$$

which maps $\mathrm{Mat}_{1,r}(V_1)\times\cdots\times\mathrm{Mat}_{1,r}(V_n) \to \mathrm{Mat}_{n,r}(V_1\cap\cdots\cap V_n)$. Observe that collection of (codimension 0 or 1) subgroups $H_1,\dots,H_n \in \mathcal{C}$ are transverse (Definition 36) if and only if the matrices coming from this map are in $\mathcal{T}$ (Definition 63).

Let $H \in \mathcal{C}$, and let $V \subseteq (\mathbb{Z}/p)^r$ be the associated linear subspace under the isomorphism $G/F \cong (\mathbb{Z}/p)^r$. By the definition of the spectrum $E_n(H)$ ([20], Definition 20)

$$\begin{aligned}\tilde{H}_*(E_n(H);\mathbb{F}_p) &\cong e_n \bigoplus_{\mathrm{Hom}(G/H,(\mathbb{Z}/p)^n)} H_*(B(\mathbb{Z}/p)^n;\mathbb{F}_p)\\ &\cong e_n \bigoplus_{\mathrm{Mat}_{n,r}(V)} H_*(B(\mathbb{Z}/p)^n;\mathbb{F}_p).\end{aligned}$$

Under the isomorphism of Equation 6.5, the product of H-summands in the Steinberg projection $\tilde{H}_*(\Phi^G\mathbf{e}_1^{\boxtimes n}B_G(\mathbb{Z}/p)^n_+) \to \tilde{H}_*(\Phi^G\mathbf{e}_nB_G(\mathbb{Z}/p)^n_+)$, namely

$$\bigotimes_{i=1}^n \tilde{H}_*(E_1(H_i);\mathbb{F}_p) \to \tilde{H}_*(E_n(H_1\cap\cdots\cap H_n);\mathbb{F}_p) \tag{6.9}$$

is given by

$$\bigotimes_{i=1}^n \left(e_1 \bigoplus_{\mathrm{Mat}_{1,r}(V_i)} H_*(B\mathbb{Z}/p;\mathbb{F}_p)\right) \to e_n \bigoplus_{\mathrm{Mat}_{n,r}(V_1\cap\cdots\cap V_n)} H_*(B(\mathbb{Z}/p)^n;\mathbb{F}_p).$$

Let $H_1,\dots,H_n \in \mathcal{C}$ be subgroups. The product $\tilde{H}_*(\Phi^G M_G(1)^{\wedge n};\mathbb{F}_p) \to \tilde{H}_*(\Phi^G M_G(n);\mathbb{F}_p)$ can be described in terms of the maps on H-summands (Definition 59)

$$\bigotimes_{i=1}^n \tilde{H}_*(M_G(1,H_i);\mathbb{F}_p) \to \tilde{H}_*(M_G(n,H_1\cap\cdots\cap H_n)). \tag{6.10}$$

Note that $M_G(1,H_i) \simeq 0$ if $d(H) \geq 2$. If the subgroups $H_1,\dots,H_n$ are transverse then by Proposition 60, the map of Equation 6.10 is given by Equation 6.9. If the subgroups $H_1,\dots,H_n$ are nontransverse then by Proposition 43, the the map of Equation 6.10 is zero. $\square$

Corollary 71. *The composition*

$$\mathbf{e}_n B_G(\mathbb{Z}/p)^n_+ \to (\mathbf{e}_1 B_G(\mathbb{Z}/p)_+)^{\wedge n} \simeq M_G(1)^{\wedge n} \to M_G(n)$$

is an isomorphism after applying the functor $\tilde{H}_*(\Phi^G(-);\mathbb{F}_p)$.

Proof. Equation 6.6 gives us a description of the first map, and Proposition 70 gives us a description of the second map. Now apply Corollary 65. □

7 Splitting of the filtration

Recall from Section 2.6 that the mod p symmetric powers of the equivariant sphere spectrum are a filtration for the equivariant Eilenberg-Maclane spectrum of $\underline{\mathbb{F}}_p$. Written below are only the stages at powers of p,

$$\Sigma^{\infty G} S^0 = \mathrm{Sp}^1(\Sigma^{\infty G} S^0) \subset \mathrm{Sp}^p(\Sigma^{\infty G} S^0) \subset \mathrm{Sp}^{p^2}(\Sigma^{\infty G} S^0) \subseteq \cdots \subseteq \mathrm{Sp}^\infty(\Sigma^{\infty G} S^0) \simeq H\underline{\mathbb{F}}_p.$$

In this section, we give a short proof of Theorem 2

Theorem 2. *Let G be any finite p group. The filtration $\{\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)\}_{n\geq 0}$ splits into its layers after smashing with $H\underline{\mathbb{F}}_p$. That is, there is an equivalence of $H\underline{\mathbb{F}}_p$-modules*

$$H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \simeq \bigvee_{n\geq 0} H\underline{\mathbb{F}}_p \wedge \Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+.$$

First, we need a lemma.

Lemma 72. *The inclusion $\Sigma^{\infty G} S^0 \to \mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)$ has a retraction after smashing with $H\underline{\mathbb{F}}_p$, namely*

$$\begin{tikzcd} \Sigma^{\infty G} S^0 \wedge H\underline{\mathbb{F}}_p \arrow[r] & \mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p \arrow[l, dashed, bend right] \end{tikzcd} .$$

Proof. The composition

$$\Sigma^{\infty G} S^0 \longrightarrow \mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \longrightarrow \mathrm{Sp}^\infty_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \simeq H\underline{\mathbb{F}}_p$$

is the unit map. Therefore, after smashing with $H\underline{\mathbb{F}}_p$, and composing with the product map $\mu : H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \to H\underline{\mathbb{F}}_p$, we have the commutative diagram

$$\begin{tikzcd} \Sigma^{\infty G} S^0 \wedge H\underline{\mathbb{F}}_p \arrow[r] \arrow[drr, "\mathrm{Id}"'] & (\mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p \arrow[r] \arrow[dr, dashed] & H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \arrow[d, "\mu"] \\ & & H\underline{\mathbb{F}}_p \end{tikzcd}$$

The dotted map provides the splitting. □

Now Theorem 2 is an immediate corollary of the following proposition.

Proposition 1. *There is a monomorphism (dotted) which splits the cofiber sequence shown.*

$$\begin{tikzcd}
H\underline{\mathbb{F}}_p \wedge \mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \arrow[r] & H\underline{\mathbb{F}}_p \wedge \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \arrow[r] & H\underline{\mathbb{F}}_p \wedge \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \arrow[l, dashed, bend right]
\end{tikzcd}$$

Proof. Let t_1 denote the retraction map of Lemma 72, i.e.

$$\begin{tikzcd}
\mathrm{Sp}^{p}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p \arrow[r] & \mathrm{Sp}^{p}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{1}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p \arrow[l, dashed, bend right, "t_1"']
\end{tikzcd} .$$

Define the map t_n by the composite in the top half of the diagram of $H\underline{\mathbb{F}}_p$-module spectra below,

$$\begin{tikzcd}
\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p \arrow[r, equal] \arrow[dd, dashed, "t_n"'] & \Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+ \wedge H\underline{\mathbb{F}}_p \arrow[d, "\subset"] \\
 & (\Sigma \mathbf{e}_1 B_G \mathbb{Z}/p_+)^{\wedge n} \wedge H\underline{\mathbb{F}}_p \arrow[d, "t_1^{\wedge n}"] \\
\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p \arrow[d] & (\mathrm{Sp}^{p}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0))^{\wedge n} \wedge H\underline{\mathbb{F}}_p \arrow[l] \arrow[d] \\
\mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p & \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p \arrow[l]
\end{tikzcd} .$$

Since t_1 is a retraction, it follows that the composite map

$$\Sigma^n \mathbf{e}_n B_G(\mathbb{Z}/p)^n_+ \wedge H\underline{\mathbb{F}}_p \to \mathrm{Sp}^{p^n}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0)/\mathrm{Sp}^{p^{n-1}}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \wedge H\underline{\mathbb{F}}_p$$

along the right and bottom of the commutative diagram, is an equivalence. Therefore, by commutativity, the composite along the left side of the diagram is an equivalence, and so t_n is a retraction. $\square$

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